General Relativity Seminars

Week 7: Lagrangian & Hamiltonian Formulations of General Relativity

Hassan Alshal

Outline

1. Lagrangian Formulation of General Relativity
2. The Initial Value Problem
3. Hamiltonian Formulation of General Relativity

Lagrangian Formulation of General Relativity

Volume Form

- Wlog and for 2×2 matrix Ω with components ${\Omega^i}_j$, we have
$$\det(\Omega) = {\Omega^0}_0{\Omega^1}_1 - {\Omega^1}_0{\Omega^0}_1 = \epsilon_{ab}{\Omega^a}_0{\Omega^b}_1, \text{ where } \epsilon_{ab} = \begin{cases} 0, & a=b \\ +1, & a=0,\ b=1 \\ -1, & a=1,\ b=0 \end{cases}.$$
- This implies $\boxed{\epsilon_{mn}\det(\Omega) = \epsilon_{ab}{\Omega^a}_m{\Omega^b}_n}$.
- For normal coordinate basis $\{y^\alpha = V^\alpha t\}$ and non-normal coordinate basis $\{x^\mu\}$,
$$dy^\alpha dy^\beta \to \left(\frac{\partial y^\alpha}{\partial x^\mu}dx^\mu\right)\left(\frac{\partial y^\beta}{\partial x^\nu}dx^\nu\right) = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}dx^\mu dx^\nu.$$
- Thus $\epsilon_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} = \epsilon_{\mu\nu}\det(\Omega) \Rightarrow \boxed{\epsilon_{\alpha\beta}dy^\alpha dy^\beta = \epsilon_{\mu\nu}\det(\Omega)dx^\mu dx^\nu}$
- Since Ω acts on $g_{\mu\nu}$, and we can construct the metric from the transformations as $g_{\mu\nu} = {\Omega^\alpha}_\mu\delta_{\alpha\beta}({\Omega_\nu}^\beta)^T$, then $\boxed{\sqrt{|\det(g)|} = \det(\Omega)}$.
- $\epsilon_{\alpha\beta}dy^\alpha dy^\beta \cong |dy^\alpha \times dy^\beta| \xRightarrow{\text{in 4-dim Lorentzian manifold}} \boxed{dV \equiv \sqrt{-\det(g)}dx^4}$.

Lagrangian Formulation of General Relativity

Interlude: The REAL Covariant Formulation of Laws of Physics

- Since volume is everything in physics (technically speaking volume in physics is not empty, it is THE stage for fields interactions that of real and virtual particles including the gravitational effects), scaling volume by a factor $\sqrt{-g}$ casts shadows on everything we developed so far! It suggests introducing the concept of "tensor density of weight W" as:
$$\boxed{\mathfrak{T}^{a\cdots}{}_{m\cdots} = (\sqrt{-g})^W \Omega^a{}_n \cdots \mho^m{}_b \cdots T^{n\cdots}{}_{b\cdots}}.$$
- The net value of W depends on the number and the nature (covariant/contravariant) of the tensor components that need to be rescaled, e.g., g_{ab} has $W = -2$ and g^{ab} has $W = +2$.
- Since dx^μ has $W = +1$, then ∂_μ has $W = -1$, i.e., $\partial_\mu \to \frac{1}{\sqrt{-g}}\partial_\mu$.
- Consequently, $\Box = \partial_\mu \partial^\mu = \partial_\mu(g^{\mu\nu}\partial_\nu) \to \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)$ with $W = 0$.
- Thus the covariant derivative needs to be rescaled too s.t.
$$\boxed{\mathfrak{T}^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s;\alpha} = T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s,\alpha} + \Gamma^{\mu_1}_{\alpha\beta}T^{\beta\cdots\mu_r}{}_{\nu_1\cdots\nu_s} + \cdots - \Gamma^{\beta}_{\alpha\nu_1}T^{\mu_1\cdots\mu_r}{}_{\beta\cdots\nu_s} - \cdots - W\Gamma^{\beta}_{\beta\alpha}T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}}$$
- Together with R being a scalar, luckily it has $W = 0$, which makes it a good candidate for being a Lagrangian density of an action that can be extremized to get Einstein's equations.

Lagrangian Formulation of General Relativity

Variation with respect to the metric

- For a full covariant theory, promote the following:

$$\boxed{\{x^\alpha\} \to \{e^a\} \quad , \quad \eta^{\mu\nu} \to g^{mn} \quad , \quad \partial \to \nabla \quad , \quad \sqrt{-\eta} \to \sqrt{-g}}.$$

- $\delta(g) = \dfrac{\partial(g)}{\partial g_{mn}} \delta g_{mn} = -g g^{mn} \delta g_{mn}$ or $\delta(g) = +g g_{mn} \delta g^{mn}$ [see Week 5 p. 5.]
- Thus $\delta(\sqrt{-g}) = \dfrac{\partial(\sqrt{-g})}{\partial(g)} \delta(g) = -\dfrac{1}{2} \dfrac{1}{\sqrt{-g}} \times -g g^{mn} \delta g_{mn} = \dfrac{1}{2} \sqrt{-g} g^{mn} \delta g_{mn}$.
- Also, $\dfrac{\delta(\delta^a{}_p)}{\delta g_{mn}} = 0 = \dfrac{\delta(g^{ab} g_{bp})}{\delta g_{mn}} = g^{ab} \dfrac{\delta(g_{bp})}{\delta g_{mn}} + g_{bp} \dfrac{\delta(g^{ab})}{\delta g_{mn}}$

  Then, $g_{bp} \dfrac{\delta(g^{ab})}{\delta g_{mn}} = -g^{ab} \dfrac{\delta(g_{bp})}{\delta g_{mn}} = -g^{ab} \dfrac{\delta(\delta^m{}_b \delta^n{}_p g_{mn})}{\delta g_{mn}} = -\delta^m{}_b \delta^n{}_p g^{ab} \dfrac{\delta(g_{mn})}{\delta g_{mn}} = -g^{am} \delta^n{}_p$

  $\therefore \delta g^{ab} = -g^{ab} \delta^n{}_p g^{bp} \delta g_{mn} = -g^{am} g^{bn} \delta g_{mn}$.
- Summary: $\boxed{\delta(\sqrt{-g}) = \dfrac{1}{2} \sqrt{-g} g^{ab} \delta g_{ab}}$ and $\boxed{\delta g^{ab} = -g^{am} g^{bn} \delta g_{mn}}$

Lagrangian Formulation of General Relativity

Einstein–Hilbert Action

- Define $S_{\text{EH}} = \dfrac{1}{16\pi}\displaystyle\int dx^4\sqrt{-g}R$.
- $\delta(\Gamma^m_{\ np}) = \frac{1}{2}g^{mq}\left[\delta(g_{qn;p}) + \delta(g_{qp;n}) - \delta(g_{np;q})\right] + \frac{1}{2}\delta g^{mq}\left[(g_{qn;p}) + (g_{qp;n}) - (g_{np;q})\right]$
  Notice that $\delta\Gamma$ is a tensor quantity.
- $\delta(\text{Riem}) = \delta(\nabla\Gamma - \nabla\Gamma) + \delta(\Gamma\Gamma - \Gamma\Gamma) = \nabla(\delta\Gamma) - \nabla(\delta\Gamma)$.
- Since $R = g^{ab}R_{ab} = g^{ab}\delta^q{}_p R^p{}_{aqb}$, then $\boxed{\delta R = \delta(g^{ab})R_{ab} + g^{ab}\delta^q{}_p\delta(R^p{}_{aqb})}$.
- $\delta R = -g^{am}g^{bn}\delta g_{mn}R_{ab} + g^{ab}\nabla_p(\delta\Gamma^p_{ab}) - \underbrace{g^{ab}\nabla_b(\delta\Gamma^p_{ap})}_{p\leftrightarrow b}$

$$= -R^{mn}\delta g_{mn} + \nabla_p(g^{ab}\delta\Gamma^p_{ab}) - \nabla_p(g^{ap}\delta\Gamma^b_{ab})$$

$$\therefore \boxed{\delta R = -R^{mn}\delta g_{mn} + \nabla_p(\mathcal{V})^p}$$

Lagrangian Formulation of General Relativity

Einstein–Hilbert Action

- $$\begin{aligned}\delta S_{\text{EH}} &= \frac{1}{16\pi}\int dx^4\left[\delta(\sqrt{-g})R + \sqrt{-g}\delta(R)\right] \\ &= \frac{1}{16\pi}\int dx^4\left[\frac{1}{2}\sqrt{-g}g^{mn}\delta g_{mn}R - \sqrt{-g}R^{mn}\delta g_{mn} + \cancel{\sqrt{-g}\nabla_p(\mathcal{V})^p}\right] \\ &= \frac{1}{16\pi}\int dx^4\sqrt{-g}\left[\frac{1}{2}g^{mn}R - R^{mn}\right]\delta g_{mn} \\ &= \frac{1}{16\pi}\int dx^4\sqrt{-g}\left[-\frac{G^{mn}}{G}\right]\delta g_{mn} \Rightarrow \boxed{\frac{1}{\sqrt{-g}}\frac{\delta S_{\text{EH}}}{\delta g_{mn}} = -\frac{G^{mn}}{16\pi G}\Big|_{\text{vac.}} = 0}\end{aligned}$$
- $$\delta S_{\text{matter}} = \delta\int dx^4\sqrt{-g}\left[\frac{1}{2}g^{ab}\nabla_a\phi\nabla_b\phi + V(\phi)\right] \xRightarrow{\text{Noether}} \boxed{T^{mn} = \frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g_{mn}}}$$
- $$\therefore \frac{1}{\sqrt{-g}}\frac{\delta(S_{\text{EH}} + S_{\text{matter}})}{\delta g_{mn}} = -\frac{G^{mn}}{16\pi G} + \frac{T^{mn}}{2} = 0 \Rightarrow \boxed{G_{ab} = 8\pi G T_{ab}}.$$
- $\delta(dx^4)$ is immanently considered in $\delta\sqrt{-g}$, see passive/active transformations.

Lagrangian Formulation of General Relativity

Conservation of Energy-Momentum Tensor

- $\delta g_{\alpha\beta} = \mathcal{L}_\xi g_{ab} = \xi^\mu g_{\alpha\beta,\mu} + \xi_{\alpha,\beta} + \xi_{\beta,\alpha} \xrightarrow{\text{covariantize}} \delta g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a$

- $$\delta S_{\text{matter}} = \int dx^4 \left[\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g_{ab}} \delta g_{ab} + \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta \phi} \delta\phi \right]$$
$$= \int dx^4 \left[\frac{1}{2}\sqrt{-g} T^{ab} (\nabla_a \xi_b + \nabla_b \xi_a) + \sqrt{-g} \frac{\delta \mathcal{L}}{\delta \phi} \mathcal{L}_\xi \phi \right]$$
$$= \int dx^4 \Big[\cancel{\frac{1}{2}\sqrt{-g}\nabla_a (T^{ab}\xi_b) + \frac{1}{2}\sqrt{-g}\nabla_b (T^{ab}\xi_a)}$$
$$- \frac{1}{2}\sqrt{-g}\nabla_a (T^{ab})\xi_b - \frac{1}{2}\sqrt{-g} \underbrace{\nabla_b (T^{ab})\xi_a}_{a \longleftrightarrow b} + \cancel{\sqrt{-g}\frac{\delta \mathcal{L}}{\delta \phi} \xi^a \nabla_a \phi} \Big]$$
$$= -\int dx^4 \sqrt{-g} \left[\nabla_b (T^{ab}) \right] \xi_b = 0$$
$$\therefore \boxed{\nabla_b (T^{ab}) = 0}$$

Normal Vectors Again!

- We have seen in wave equation treatment that $G_{ab} \sim R_{ab} \sim g^{cd} g_{ac,bd}$ in $\mathcal{M}_{d=4}$ which is a 2nd order differential equation. Just like how we need $x(t=0)$ and $\dot{x}(t=0)$ to solve $\ddot{x}(t)$, we also need $\mathfrak{g}_{ij}$ and $\partial_t \mathfrak{g}_{ij}$ to solve G_{ab}, where $\mathfrak{g}_{ij}$ is the "induced metric of hypersurface" $\Sigma_{d=3}$. We define $\hat{N}^a$ as the normal to $\Sigma_{d=3}$.
- We define the signature of the normal $\hat{N}_a \hat{N}^a = \pm 1 = \mathfrak{s}$ depending on whether $\Sigma_{d=3}$ is timelike or spacelike respectively.
- Side note: if $\Sigma_{d=3}$ is lightlike, then $\hat{N}_a \hat{N}^a = 0$, which means $\hat{N} \in T_p \Sigma_{d=3}$!!!!!
- Also, $\mathfrak{g}_{ij} \hat{N}^j = 0 \Rightarrow \boxed{\mathfrak{g}_{ij} = \Omega^a{}_i \Omega^b{}_j g_{ab} \mp \hat{N}_i \hat{N}_j} \xRightarrow{\text{normal coordinate}} \boxed{\mathfrak{g}_{ij} = \eta_{ij} \mp \hat{N}_i \hat{N}_j}$

  [This should remind you of the photon projection operator in QFT.]
- With the former information one can <u>prove</u> the following:

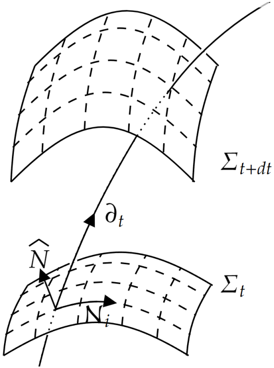
  i. $\mathfrak{g}^i{}_k \mathfrak{g}^k{}_j = \mathfrak{g}^i{}_j$.

  ii. $\forall X^a \in T_p\mathcal{M}$, we have $X^a = X^a_{||} + X^a_{\perp}$ s.t. $X^a_{||} = \mathfrak{g}^a{}_b X^b$ and $X^a_{\perp} = \pm \hat{N}_b X^b \hat{N}^a$.

  iii. $\forall X^i,\ Y^j \in T_p\Sigma$, we have $\mathfrak{g}_{ij} X^i Y^j = g_{ab} X^a Y^b$.

The Initial Value Problem

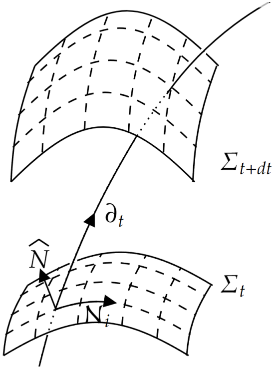
Spacetime Embedding & Foliation



- $g_{\mu\nu} = \left[\begin{array}{c|c} g_{tt} & \mathcal{N}_i \\ \hline \mathcal{N}_i & \mathfrak{g}_{ij} \end{array}\right]$, and $(\mathfrak{g}_{ij})^{-1} = \mathfrak{g}^{ij}$ for non-lightlike Σ, together with $\mathcal{N}_i = g_{ti} = e_t \otimes e_i$. Needless to say e_t is NOT necessarily timelike vector.
- Notice that if $e_t \perp \Sigma_t$, then $\mathcal{N}_i = 0$ as $\{e_i\} \subset T_p\Sigma_t$. This means from perspective of an observer on $T_p\Sigma_t$ the "vector" $\mathcal{N}_i$ is defined in terms of $\{e_i\}$, and thus $\mathcal{N}_i \in \Sigma_t$. Therefore, we can say that $\mathcal{N}_i$ measures how e_t is "shifted" from being orthogonal to Σ_t.
- In the language of diffeomorphisms $\mathfrak{g}_{ij} = \psi^* g_{\mu\nu}$ for $\psi : T_p\Sigma_t \rightarrow T_p\mathcal{M}$, which is the "embedding function" that describes "manifold foliation".
- Coordinatewise and for $T_p\Sigma_t$ basis $\{y^i\}$, since $e_t \equiv \partial_t$, then for $T_p\mathcal{M}$ basis $\{x^\mu\}$ we get $\mho^\mu{}_i = \nabla_{e^i} e^\mu$. But $\mho^\mu{}_i$ is NOT a square matrix, WHY? $(\mho^\mu{}_i)^{-1} = \Omega^i{}_\mu$ is ok!
- Decompose $\mho^\mu{}_i$ into its i components like how $\Gamma^\mu_{\nu\rho}$ is decomposed into μ components.
- $\mho^\mu{}_i \equiv E^\mu{}_i$ (or $\Omega^i{}_\mu \equiv E^i{}_\mu$) are called "Cartan tetrads" and work fine as vectors for $T_p\Sigma_t$.

The Initial Value Problem

Lapse & Shift functions



- Then, we define the normal to $T_p\Sigma_t$ as $\hat{N}_\mu E^\mu_{\ i} = 0$. And if $\partial_t \perp T_p\Sigma_t$, then $\hat{N}_\mu \equiv \partial_t = (\hat{N}_0, 0, 0, 0)$ as $\hat{N}_\mu \in T_p\mathcal{M}$.
- Also for "normal" $\{y^i\}$ basis, $e^\mu = E^\mu_{\ i} e^i = \delta^\mu_{\ i} e^i \Rightarrow \boxed{g_{\mu\nu} = \delta^i_{\ \mu}\delta^j_{\ \nu} e_i e_j}$. This means that $\hat{N}_\mu E^\mu_{\ i} = \hat{N}_\mu \delta^\mu_{\ i} = 0$ as $\mu \neq i$ in such coordinate system.
- We saw in TNB frame that $\hat{N} = \partial\hat{T}/|\partial\hat{T}|$. To promote it in a Lorentian manifold, then $\boxed{\hat{N}_\mu = \left.\frac{\nabla_\mu(e_\alpha e^\alpha)}{\sqrt{g_{\mu\nu}\nabla^\mu(e_\alpha e^\alpha)\nabla^\nu(e_\alpha e^\alpha)}}\right|_{\alpha\neq i} \Rightarrow \hat{N}_\mu \hat{N}^\mu = \mathfrak{s}}$. As $T_p\Sigma_t$ is spacelike, then $\hat{N}_\mu \hat{N}^\mu = -1$. Usually for spherical diagonal metric $e_\alpha e^\alpha = g_{00}$.
- As ∇_t is not necessarily normal, then define $\hat{N}_0 = \mathfrak{n} \Rightarrow \hat{N}(e^t) = \mathfrak{s}/\mathfrak{n} = -1/\mathfrak{n}$ such that $g^{00} = e^t e^t = (-1)\frac{\hat{N}\hat{N}}{\mathfrak{n}\mathfrak{n}} = \mathfrak{s}(\mathfrak{n})^{-2} \Rightarrow \boxed{g_{00} = -\mathfrak{n}^2}$
- Lapse $\mathfrak{n}$ defines $\mathfrak{s}\frac{d\tau}{dt}$ while moving along $\hat{N}$ from $T_p\Sigma_t$ to $T_p\Sigma_{t+dt}$.

The Initial Value Problem

Contravariant Induced Metric & Drag Speed of Space

- $g_{\mu\nu} = \left[\begin{array}{c|c} g_{tt} & \mathcal{N}_i \\ \hline \mathcal{N}_i & \mathfrak{g}_{ij} \end{array}\right] = \left[\begin{array}{c|c} \mathfrak{sn}^2 & \mathcal{N}_i \\ \hline \mathcal{N}_i & \mathfrak{g}_{ij} \end{array}\right].$
- $M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \Rightarrow M^{-1} = \left[\begin{array}{c|c} (A-BCD^{-1})^{-1} & -(A-BCD^{-1})^{-1}BD^{-1} \\ \hline -CD^{-1}(A-BCD^{-1})^{-1} & D^{-1}+CD^{-1}(A-BCD^{-1})^{-1}BD^{-1} \end{array}\right]$
- $g^{\mu\nu} = \left[\begin{array}{c|c} \mathfrak{sn}^{-2} & -\mathfrak{sn}^{-2}\mathcal{N}^i \\ \hline -\mathfrak{sn}^{-2}\mathcal{N}^i & \mathfrak{g}^{ij}+\mathfrak{sn}^{-2}\mathcal{N}^i\mathcal{N}^j \end{array}\right]$
- As we see despite that $\mathfrak{g}_{ij} = g_{ij}$, $\boxed{\mathfrak{g}^{ij} \neq g^{ij}}$, i.e., The contravariant spatial components of the surface metric are not the same as those of the bulk metric. But if $g_{\mu\nu}$ is diagonal $\xRightarrow{\mathcal{N}^i=0} g^{ij} = \mathfrak{g}^{ij}$.
- This means $\hat{N}^\mu = g^{\mu\nu}\hat{N}_\nu = [g^{tt}, g^{ti}]\hat{N}_0 = [\mathfrak{sn}^{-2}, -\mathfrak{sn}^{-2}\mathcal{N}^i]\cdot\mathfrak{sn} = [\mathfrak{n}^{-1}, -\mathfrak{n}^{-1}\mathcal{N}^i] \neq 0$
- Since $\hat{N}^\mu = \hat{N}^t e_t + \hat{N}^i e_i = \mathfrak{n}^{-1}e_t - \mathfrak{n}^{-1}\mathcal{N}^i e_i$, then $\boxed{\nabla_t \equiv e_t = \mathfrak{n}\hat{N} + \mathcal{N}^i e_i}$
- To move along ∇_t from $T_p\Sigma_t$ to $T_p\Sigma_{t+dt}$ we need to lapse along $\hat{N}$ then shift along e_i.
- $g^{ij} = \mathfrak{g}^{ij} + \mathfrak{sn}^{-2}\mathcal{N}^i\mathcal{N}^j$ reminds us of $\mathfrak{g}_{ij} = \Omega^a{}_i\Omega^b{}_j g_{ab} \mp \hat{N}_i\hat{N}_j$. So, if we define $V = \sqrt{\mathcal{N}_i\mathcal{N}^i}$ as the "Drag Speed of Space", then $\boxed{-(g_{tt})^{-1/2}e_t\hat{N} = (1-V^2/\mathfrak{n}^2)^{-1/2} := \gamma_V}$. [What does it mean if $\mathfrak{n} < \mathcal{N}_i$?!]

Extrinsic Curvature

- Let $Y \in T_p\Sigma_t$ and $X \in T_p\mathcal{M}$. If you parallel transport $\hat{N}$ along X, i.e., $X^a \nabla_a \hat{N} = 0$, then $X(\hat{N}_b Y^b) = \cancel{X(\hat{N}_b) Y^b} + \hat{N}_b X(Y^b) = \hat{N}_b X^a \nabla_a (Y^b) = 0$. So $\hat{N} \perp X^a \nabla_a (Y)$ or $\nabla_X (Y) \in T_p\Sigma_t$, i.e., only parallel components survive.
- Define $\boxed{\mathcal{K}(X,Y) = -\hat{N}_a (\nabla_{X_{||}} Y_{||})^a}$ as the "Extrinsic Curvature".
- From properties of $X_{||}$ and $Y_{||}$, and as $\mathcal{K}_{ab} = \mathcal{K}_{ba}$, then one can prove that $\mathcal{K} = X^a_{||} Y^b_{||} \nabla_a \hat{N}_b = X^a_{||} Y^b_{||} \mathfrak{g}^c{}_a \mathfrak{g}^d{}_b \nabla_c \hat{N}_d \Rightarrow \boxed{\mathcal{K}_{ab} = \mathfrak{g}^c{}_a \nabla_c \hat{N}_b}$.
- The last finding, together with the symmetric nature, helps defining $\boxed{\mathcal{K}_{ab} = \frac{1}{2} \mathcal{L}_{\hat{N}} \mathfrak{g}_{ab}}$.

The Initial Value Problem

Tensors on Hypersurfaces & Constraints on Einstein's Equations

- Using the diffeomorphism $\psi : \Sigma \to \mathcal{M}$, we can pull-back $\Gamma^{\mu}_{\nu\rho} \to \gamma^{i}_{jk}$, where γ^{i}_{jk} is the Christoffel symbol of Σ. And to avoid torsion, $\gamma^{i}_{ik} = \gamma^{i}_{kj}$.
- This means we can impose metricity on $\mathfrak{g}_{ij}$ using the induced covariant derivative $\mathcal{D}$ s.t. $\mathcal{D}_k(\mathfrak{g}_{ij}) = \partial_k \mathfrak{g}_{ij} - \gamma^{l}_{ki}\mathfrak{g}_{lj} - \gamma^{l}_{jk}\mathfrak{g}_{il} = 0$.
- $\mathcal{D}$ can be seen as the "projection" of ∇ on Σ, i.e., $\boxed{\mathcal{D}_i = \mathfrak{g}^{j}{}_{i}\nabla_j}$.
- With help of all these findings, one can prove that the induced Riemann tensor on Σ is

$$\boxed{\mathbb{R}^{i}{}_{jkl} = \mathfrak{g}^{i}{}_{a}\mathfrak{g}^{b}{}_{j}\mathfrak{g}^{c}{}_{k}\mathfrak{g}^{d}{}_{l}R^{a}{}_{bcd} + \mathfrak{s}(\mathcal{K}^{i}{}_{k}\mathcal{K}_{lj} - \mathcal{K}^{i}{}_{l}\mathcal{K}_{kj})}.$$

- After obtaining the induced $\mathbb{R}_{ij}$ and $\mathbb{R}$, one can prove that

$$\boxed{G_{ab}\hat{N}^a\hat{N}^b = R_{ab}\hat{N}^a\hat{N}^b - \frac{1}{2}R = \frac{1}{2}(\mathbb{R} - \mathcal{K}_{ab}\mathcal{K}^{ab} + \mathcal{K}^2) = 8\pi G T_{tt}} \text{ as energy constraint.}$$

$$\boxed{\mathfrak{g}^{k}{}_{i}G^{ij}\hat{N}_j = \mathcal{D}_i\mathcal{K}^{ki} - \mathcal{D}^k\mathcal{K} = 8\pi G\mathfrak{g}^{k}{}_{i}T^{ij}\hat{N}_j} \text{ as momentum constraint.}$$

The Problem of Time in ADM formalism of General Relativity

- $\mathcal{K}_{ij} = -\frac{1}{2}\mathcal{L}_{\hat{N}}\mathbb{g}_{ij}$ indicates $\partial_t \mathbb{g}_{ij}$ is important to consider in δS_{EH}. So after applying the induced derivatives and induced Christoffel symbols, one gets $\dot{\mathbb{g}}_{ij} = -2\mathbb{n}\mathcal{K}_{ij} + 2\mathcal{D}_{(i}\mathcal{N}_{j)}$.
- The space component of S_{EH} is defined $S_\Sigma = \int (\mathbb{R} + \mathcal{K}_{ij}\mathcal{K}^{ij} - \mathcal{K}^2)\mathbb{n}\sqrt{\mathbb{g}}dx^3$
- Then, we can define a “conjugate momentum” as $\Pi^{ij} = \dfrac{\partial S_\Sigma}{\partial \dot{\mathbb{g}}_{ij}}$.
- Then, $$\mathcal{H} = \int (\Pi^{ij}\dot{\mathbb{g}}_{ij} - \mathcal{L}_\Sigma)dx^3$$ $$= 16\pi \int \left(\Pi^{ij}\Pi_{ij} - \frac{1}{\mathbb{n}-1}\Pi^2\right)\frac{\mathbb{n}}{\sqrt{\mathbb{g}}}dx^3 - 2\int \mathcal{D}_i\left(\frac{\Pi^{ij}}{\sqrt{\mathbb{g}}}\right)\mathcal{N}^i\sqrt{\mathbb{g}}dx^3 - \frac{1}{16\pi}\int \mathbb{R}\mathbb{n}\sqrt{\mathbb{g}}dx^3$$
- We do not have terms like $\dot{\mathbb{n}}$ or $\dot{\mathcal{N}}^i$, which means $\Pi_{\mathbb{n}} = \Pi_{\mathcal{N}^i} = 0$ is another constraint.
- Then for $\delta\mathcal{H} = \int \left(\dfrac{\partial \mathcal{H}}{\partial \mathbb{n}}\delta\mathbb{n} + \dfrac{\partial \mathcal{H}}{\partial \mathcal{N}^i}\delta\mathcal{N}^i\right)dx^3$
- Due to the constraints found before, it terns out that $\dfrac{\partial \mathcal{H}}{\partial \mathbb{n}} = \dfrac{\partial \mathcal{H}}{\partial \mathcal{N}^i} = 0$, i.e., No Hamiltonian!

Thank You!