General Relativity Seminars

Week 6: Curvature, Einstein's field equations & gravitational approximations

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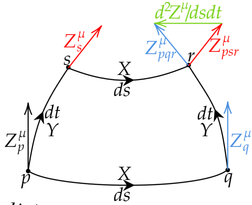
Outline

1. Intrinsic Curvature
2. Einstein Field Equations
3. Weak Gravity Source

Parallel Transport

- Parallel transport of a vector Z along the curves means $\nabla_X Z = \nabla_Y Z = 0$. Componentwise in $\{\partial_\mu\}$ basis this reads $X^\nu Z^\mu{}_{,\nu} + \Gamma^\mu_{\rho\nu} Z^\rho X^\nu = 0 \Rightarrow \boxed{X^\nu Z^\mu{}_{,\nu} = -\Gamma^\mu_{\rho\nu} Z^\rho X^\nu}$.
- $\dfrac{d}{ds} = X^\nu \partial_\nu$ AND $\dfrac{d}{dt} = Y^\nu \partial_\nu$. Also, expand up to $\mathcal{O}(d^3)$.
- $Z^\mu_q = Z^\mu_p + \dfrac{d}{ds} Z^\mu_p ds + \cdots = Z^\mu_p - \Gamma^\mu_{\rho\nu} Z^\rho_p X^\nu ds + \cdots$
- $$\begin{aligned} Z^\mu_{pqr} &= Z^\mu_q + \frac{d}{dt}\left(Z^\mu_q\right) dt = Z^\mu_p - \Gamma^\mu_{\rho\nu} Z^\rho_p X^\nu ds + \frac{d}{dt}\left(Z^\mu_p - \Gamma^\mu_{\rho\nu} Z^\rho_p X^\nu ds\right) dt + \cdots \\ &= Z^\mu_p - \Gamma^\mu_{\rho\nu} Z^\rho_p X^\nu ds + \left(Z^\mu_{p,\sigma} Y^\sigma - \Gamma^\mu_{\rho\nu,\sigma} Y^\sigma Z^\rho_p X^\nu ds - \Gamma^\mu_{\rho\nu} Z^\rho_{p,\sigma} Y^\sigma X^\nu ds\right) dt + \cdots \\ &= Z^\mu_p - \Gamma^\mu_{\rho\nu} Z^\rho_p X^\nu ds - \Gamma^\mu_{\rho\sigma} Z^\rho_p Y^\sigma dt - \Gamma^\mu_{\rho\nu,\sigma} Y^\sigma Z^\rho_p X^\nu dsdt + \underbrace{\Gamma^\mu_{\rho\nu}\Gamma^\rho_{\lambda\sigma} Z^\lambda_p Y^\sigma X^\nu dsdt}_{\lambda \longleftrightarrow \rho} + \cdots \end{aligned}$$
- The extra terms containing $X^\nu{}_{,\sigma} Y^\sigma$ and $Y^\nu{}_{,\sigma} X^\sigma$ in Z^μ_{pqr} and Z^μ_{psr} cancel each other.
- $\boxed{\therefore Z^\mu_{pqr} = Z^\mu_p - \Gamma^\mu_{\rho\nu} Z^\rho_p X^\nu ds - \Gamma^\mu_{\rho\sigma} Z^\rho_p Y^\sigma dt - \Gamma^\mu_{\rho\nu,\sigma} Y^\sigma Z^\rho_p X^\nu dsdt + \Gamma^\mu_{\lambda\nu}\Gamma^\lambda_{\rho\sigma} Z^\rho_p Y^\sigma X^\nu dsdt}$
- $\boxed{\& \ Z^\mu_{psr} = Z^\mu_p - \Gamma^\mu_{\rho\nu} Z^\rho_p Y^\nu dt - \Gamma^\mu_{\rho\sigma} Z^\rho_p X^\sigma ds - \Gamma^\mu_{\rho\nu,\sigma} X^\sigma Z^\rho_p Y^\nu dtds + \Gamma^\mu_{\lambda\nu}\Gamma^\lambda_{\rho\sigma} Z^\rho_p X^\sigma Y^\nu dtds}$



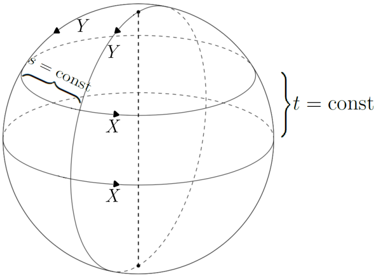
Intrinsic Curvature

Riemann Curvature Tensor

- $$Z^{\mu}_{pqr} = \cancel{Z^{\mu}_{p}} - \cancel{\Gamma^{\mu}_{\rho\nu} Z^{\rho}_{p} X^{\nu} ds} - \cancel{\Gamma^{\mu}_{\rho\sigma} Z^{\rho}_{p} Y^{\sigma} dt} - \Gamma^{\mu}_{\rho\nu,\sigma} Y^{\sigma} Z^{\rho}_{p} X^{\nu} ds dt + \Gamma^{\mu}_{\lambda\nu} \Gamma^{\lambda}_{\rho\sigma} Z^{\rho}_{p} Y^{\sigma} X^{\nu} ds dt$$
  $$Z^{\mu}_{psr} = \cancel{Z^{\mu}_{p}} - \cancel{\Gamma^{\mu}_{\rho\nu} Z^{\rho}_{p} Y^{\nu} dt} - \cancel{\Gamma^{\mu}_{\rho\sigma} Z^{\rho}_{p} X^{\sigma} ds} - \underbrace{\Gamma^{\mu}_{\rho\nu,\sigma} X^{\sigma} Z^{\rho}_{p} Y^{\nu} dt ds + \Gamma^{\mu}_{\lambda\nu} \Gamma^{\lambda}_{\rho\sigma} Z^{\rho}_{p} X^{\sigma} Y^{\nu} dt ds}_{\sigma \longleftrightarrow \nu}$$
- $$\lim_{\substack{ds\to 0\\ dt\to 0}} \frac{d^2}{ds dt}\left(Z^{\mu}\right) = \lim_{\substack{ds\to 0\\ dt\to 0}} \frac{Z^{\mu}_{pqr} - Z^{\mu}_{psr}}{ds dt}$$
  $$= \left(\Gamma^{\mu}_{\rho\sigma,\nu} - \Gamma^{\mu}_{\rho\nu,\sigma} + \Gamma^{\mu}_{\lambda\nu}\Gamma^{\lambda}_{\rho\sigma} - \Gamma^{\mu}_{\lambda\sigma}\Gamma^{\lambda}_{\rho\nu}\right) X^{\nu} Y^{\sigma} Z^{\rho}_{p}$$
- Last result defines the "Riemann Curvature Tensor" $\boxed{\lim_{\substack{ds\to 0\\ dt\to 0}} \frac{d^2}{ds dt}\left(Z^{\mu}\right) := R^{\mu}{}_{\rho\sigma\nu} X^{\nu} Y^{\sigma} Z^{\rho}}$, that is the "intrinsic curvature tensor" which measures the variation in vectors due to parallel transport on surfaces defined by two geodesics $\nabla_X Z = 0$ and $\nabla_Y Z = 0$.
- In basis-independent system, $\boxed{R(X,Y)Z := \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) Z}$
- Or $\boxed{\frac{d}{ds}\left(\frac{d}{dt}\left(\psi_s^{-1} \circ \psi_t^{-1} \circ \psi_s \circ \psi_t (Z)\right)\right)\Big|_{s=t=0} := R(X,Y)Z}$ for $\psi_{x_t} : T_{x_0}\mathcal{M} \to T_{x_t}\mathcal{M}$.

Intrinsic Curvature

Geodesic Deviation



- In the past definition of Riemann tensor, $\nabla_{[X,Y]}Z$ term disappears in $\{\partial_\mu\}$ basis as $[\partial_\mu, \partial_\nu]Z = 0$.
- Also, $R(X,Y)Z = -R(Y,X)Z$, i.e., $\boxed{R^\mu{}_{\rho(\sigma\nu)} = 0}$.
- Additionally, one can prove that Riemann tensor is linear, i.e., $\boxed{R(fX,Y)Z = fR(X,Y)Z}$ and $\boxed{R(X,Y)(fZ) = fR(X,Y)Z}$.
- Moreover, one can exploit the definition of Riemann tensor in terms of Christoffel symbols to prove $\boxed{R^\mu{}_{[\rho\sigma\nu]} = 0}$ and $\boxed{\nabla_{[\tau}R^\mu{}_{|\rho|\sigma\nu]} = R^\mu{}_{\rho[\sigma\nu;\tau]} = 0}$ which is the "Bianchi identity".
- Furthermore for $R_{\mu\rho\sigma\nu} = g_{\mu\lambda}R^\lambda{}_{\rho\sigma\nu}$, we have $\boxed{R_{\mu\rho\sigma\nu} = R_{\sigma\nu\mu\rho}}$ and $\boxed{R_{(\mu\rho)\sigma\nu} = 0}$.
- Since $[X,Y] = 0$, then $R(X,Y)X = \nabla_X\nabla_Y X - \nabla_Y\cancel{\nabla_X X} = \nabla_X\nabla_X Y$. Therefore, $\boxed{R(X,Y)X = \nabla_X\nabla_X Y}$ defines "geodesic deviations". We use this result to prove cylinders, in contrary with spheres, do NOT have intrinsic curvature despite having extrinsic one.

[For a torus, is $R^i_{jkl} = 0$? NO! This is why topological features are essential in the definition of manifolds.]

Ricci Decomposition

- Define $\mathrm{Ric}(Y,Z) \equiv \mathrm{tr}(X \to R(X,Y)Z) = \langle R(e_a,Y)Z, e_a\rangle$.
- $g_{ae}g^{ec}R^a{}_{bcd} = \delta_a{}^c R^a{}_{bcd} \equiv \boxed{R^a{}_{bad} \equiv R_{ab}}$. with symmetric property $R_{ab} = R_{ba}$.
- In coordinate basis $R_{\rho\nu} = \Gamma^\mu_{\rho\mu,\nu} - \Gamma^\mu_{\rho\nu,\mu} + \Gamma^\mu_{\lambda\nu}\Gamma^\lambda_{\rho\mu} - \Gamma^\mu_{\lambda\mu}\Gamma^\lambda_{\rho\nu}$.
- Define $R \equiv \mathrm{tr}(\mathrm{tr}(X \to R(X,Y)Z)) = \langle R(e_a,e_b)e_b, e_a\rangle$.
- $\boxed{R \equiv g^{ab}R_{ab}}$.
- In coordinate basis $R = g^{\rho\nu}\left(\Gamma^\mu_{\rho\mu,\nu} - \Gamma^\mu_{\rho\nu,\mu} + \Gamma^\mu_{\lambda\nu}\Gamma^\lambda_{\rho\mu} - \Gamma^\mu_{\lambda\mu}\Gamma^\lambda_{\rho\nu}\right)$.
- Define the traceless Ricci tensor $Z_{ab} = R_{ab} - \frac{1}{d}Rg_{ab} = g^{ad}E_{abcd}$, i.e., $Z^a{}_a = 0$, s.t.

$$\left.\begin{aligned} S_{abcd} &= \frac{R}{d(d-1)}\left(g_{ad}g_{bc} - g_{ac}g_{bd}\right) \\ E_{abcd} &= \frac{1}{d-2}\left(Z_{ad}g_{bc} - Z_{bd}g_{ac} - Z_{ac}g_{bd} + Z_{bc}g_{ad}\right) \end{aligned}\right\} \begin{aligned} & W_{abcd} = R_{abcd} - S_{abcd} - E_{abcd} \\ & g^{ad}W_{abcd} = 0 \end{aligned}$$

- $$\begin{aligned} R_{abcd}R^{abcd} &= S_{abcd}S^{abcd} + E_{abcd}E^{abcd} + W_{abcd}W^{abcd} \\ &= W_{abcd}W^{abcd} + \frac{4}{d-2}R_{ab}R^{ab} - \frac{2}{(d-1)(d-2)}R^2, \end{aligned}$$

  which is "Kretschmann scalar".

Einstein Field Equations

Einstein Tensor

- R by itself is not enough since gravity needs to be described by T^{ab}.
- ~~$R_{ab} = \alpha T_{ab}$~~ as $\nabla_a R^{ab} \neq 0$ but $\nabla_a T^{ab} = 0$.
- Also ~~$R_{abcd} = \alpha T_{ab} T_{cd}$~~ as it means $R^{\text{vac.}}_{abcd} = 0$, which is generally incorrect, e.g., black holes.
- Define $\boxed{G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}} \Rightarrow \nabla_a G^{ab} = 0$ as one can prove $\nabla_a R^{ab} = \frac{1}{2}\nabla^b R$.
- $\boxed{G_{ab} \equiv \frac{8\pi G}{c^4} T_{ab}}$ which means:
  "Matter tells spacetime how to curve, and curved spacetime tells matter how to move."
  - J. A. Wheeler.
- Also, one can prove that $R_{ab} = \frac{8\pi G}{c^4}\left(T_{ab} - \frac{1}{2} T g_{ab}\right)$, where $T = T^a_a = g_{ab} T^{ab}$.
- Lovelock theorem: $G_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab}$, where Λ is "Einstein's biggest blunder", the cosmological constant.

Weak Gravity Source

1st Order Approximation

From now on, we will indulge in a special system of units that set $c = 1$, which is tempting and dangerous sometimes!

- $g_{ab} = \eta_{ab} + h_{ab} \Rightarrow g^{ab} = \eta^{ab} - h^{ab}$
- $\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}(\eta^{\mu\sigma} - h^{\mu\sigma})(h_{\sigma\nu,\rho} + h_{\sigma\rho,\nu} - h_{\nu\rho,\sigma}) \Rightarrow \boxed{\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}(h_{\sigma\nu,\rho} + h_{\sigma\rho,\nu} - h_{\nu\rho,\sigma})}$
- $R_{\mu\nu\rho\sigma} = g_{\mu\tau}R^{\tau}{}_{\nu\rho\sigma} = (\eta_{\mu\tau} + h_{\mu\tau})(\Gamma^{\tau}_{\nu\rho,\sigma} - \Gamma^{\tau}_{\nu\sigma,\rho} + \Gamma^{\lambda}_{\nu\rho}\Gamma^{\tau}_{\lambda\sigma} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\tau}_{\lambda\rho})$
  $= \eta_{\mu\tau}(\Gamma^{\tau}_{\nu\rho,\sigma} - \Gamma^{\tau}_{\nu\sigma,\rho})$ as other terms contain higher orders.
- $R_{\mu\nu\rho\sigma} = \frac{1}{2}\eta_{\mu\tau}\eta^{\tau\lambda}(\cancel{h_{\lambda\nu,\rho\sigma}} + h_{\lambda\rho,\nu\sigma} - h_{\nu\rho,\lambda\sigma} - \cancel{h_{\lambda\nu,\sigma\rho}} - h_{\lambda\sigma,\nu\rho} + h_{\nu\sigma,\lambda\rho})$
  $= h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\nu\sigma,\mu\rho} - h_{\mu\rho,\nu\sigma}$
- After introducing $h \equiv h^{\mu}{}_{\mu}$ one can prove

  $$\boxed{R_{\mu\nu} = \eta^{\alpha\beta}R_{\alpha\mu\beta\nu} = \partial^{\rho}\partial_{[\mu}h_{\nu]\rho} - \frac{1}{2}\Box h_{\mu\nu} - \frac{1}{2}\partial_{\mu}\partial_{\nu}h}.$$
- And for $\boxed{R = \eta^{\mu\nu}R_{\mu\nu} = \partial^{\rho}\partial^{\sigma}h_{\rho\sigma} - \Box h}$.

Weak Gravity Source

Killing Vector & Gauge Condition

- $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = 8\pi G T_{\mu\nu}$
- Introduce $\bar{h}_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu}h$, $\bar{h}^{\mu\nu} = h^{\mu\nu} - \eta^{\mu\nu}h$, $\bar{h} = \bar{h}^{\mu}{}_{\mu} = -h$, s.t.
- $G_{\mu\nu} = \partial^{\rho}\partial_{[\mu}\bar{h}_{\nu]\rho} - \frac{1}{2}\Box\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\partial^{\rho}\partial^{\sigma}\bar{h}_{\rho\sigma} = 8\pi G T_{\mu\nu}$.
- $(\psi_{-t})_{*}g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{L}_{\xi}\eta_{\mu\nu} + \cdots = \eta_{\mu\nu} + h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$
- $\boxed{h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}}$ the de Donder-Einstein-Lorenz-Weyl gauge.
  It is similar to Lorenz gauge of EM $A_{\mu} \to A_{\mu} + \theta_{,\mu}$ with condition $A_{\mu,\mu} = 0$
- Similarly, $\bar{h}_{\mu\nu,\mu} = 0$ s.t. $\Box\xi_{\mu} = 0$ which is not a wave equation just like $\Box\theta = 0$.
- $\boxed{\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}}$

Weak Gravity Source

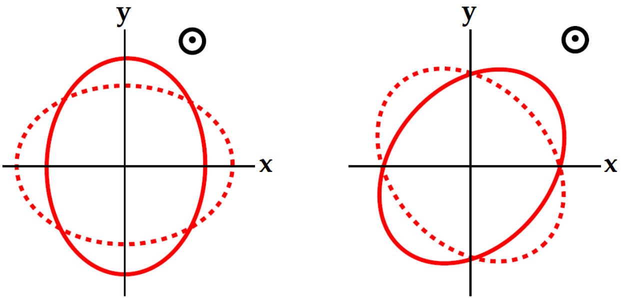
Solutions in vacuum

Gravitational Modes

$$\Box \bar{h}_{\mu\nu} = 0$$

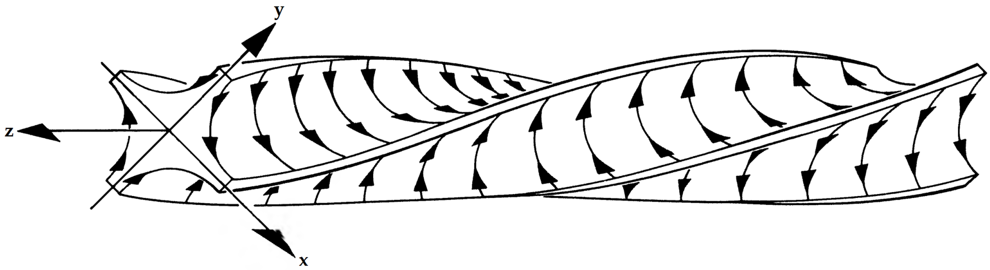
$$\bar{h}_{\mu\nu} = H_{\mu\nu} e^{ik_\alpha x^\alpha}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$



Amplitude$\sim 10^{-20} m$. You can stack 10^5 amplitudes within the radius of a proton! See Felix Pirani's "The Sticky Bead Argument" in Conference on the Role of Gravitation in Physics at the University of North Carolina, Chapel Hill, 1957.

Weak Gravity Source



C. W. Misner, K .S. Thorne, J. A. Wheeler, Gravitation, W. H. Freeman & Company, (1973), p. 1022

Birkhoff–Israel theorem

- In the framework of general relativity Birkhoff–Israel theorem states that a spherically symmetric solution of $G_{ab} = 0 \Leftrightarrow$ static and asymptotically flat.

- However, in the framework of Newtonian gravity, the shell theory states that a static and asymptotically flat solution $\overset{\not\Rightarrow}{\Leftarrow}$ spherically symmetric of $\ddot{x}$, Why?

- It means the best choice for g_{00} and g_{11} is to select exponential functions, say

$$\boxed{g_{\mu\nu} = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2}.$$

- One can use this metric to obtain $\Gamma^{\lambda}_{\mu\nu}$ components, and hence those of $R_{\mu\nu}$.

Weak Gravity Source

Schwarzschild's Solution

- Since $R_{\mu\nu} = 0$ in vacuum, then for $' \equiv \frac{d}{dr}$ we have
- $R_{11} = A'' + 2(A')^2 - A'(A' + B') - \frac{2}{r} B'$.
- $R_{00} = -[A'' + 2(A')^2] e^{2(A-B)} + A' e^{2(A-B)} (A' + B') - \frac{2}{r} A' e^{2(A-B)}$.
- $R_{11} + e^{2(B-A)} R_{00} = -\frac{2}{r}(A' + B') = 0 \Rightarrow A + B = \text{const.}$
- Asymptotic flatness demands $A, B \xrightarrow{r\to\infty} 0$ s.t. $g_{\mu\nu}(r \to \infty) \equiv \eta_{\mu\nu} \Rightarrow \boxed{A = -B}$.
- $R_{22} = -1 + r(A' - B') e^{-2B} + e^{-2B} = 0 \Rightarrow 1 = 2rA' e^{2A} + e^{2A} = (re^{2A})'$.

  The solution of $1 = (re^{2A})'$ is $re^{2A} - r = \text{const.}$ or $\boxed{e^{2A} = 1 - \frac{\text{const.}}{r}}$.

  $$h_{00} = g_{00} - \eta_{00} = -\left(1 - \frac{\text{const.}}{r}\right) - (-1) = \frac{\text{const.}}{r} \underset{\text{Weak}}{\approx} \frac{2GM}{r}$$

- $$\boxed{g_{\mu\nu} = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2}$$

Weak Gravity Source

Energy Conditions from the Metric

- Remember when we extremized $\int ds = \int dt \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$ to get the geodesics?
- Extremize ds^2 to get dynamical conditions for objects moving along geodesics, i.e., for $\dot{x}^\mu = dx^\mu/d\tau$, apply variational approach on
- $$L = -\left(1-\frac{2GM}{r}\right)(\dot{x}^t)^2 + \left(1-\frac{2GM}{r}\right)^{-1}(\dot{x}^r)^2 + r^2(\dot{x}^\theta)^2 + r^2\sin^2\theta(\dot{x}^\phi)^2$$
$$= -\left(1-\frac{2GM}{r}\right)(\dot{t})^2 + \left(1-\frac{2GM}{r}\right)^{-1}(\dot{r})^2 + r^2(\dot{\theta})^2 + r^2\sin^2\theta(\dot{\phi})^2.$$
- For for orbiting object with mass $m = 1$, Kepler laws remind us $\theta = \pi/2$, i.e.,

Weak Gravity Source

Energy Conditions from the Metric

- $L = -\left(1 - \frac{2GM}{r}\right)(\dot{t})^2 + \left(1 - \frac{2GM}{r}\right)^{-1}(\dot{r})^2 + r^2(\dot{\phi})^2$
- Variational ϕ reads $0 = \frac{dL}{d\phi} - \frac{d}{d\tau}(\frac{dL}{d\dot{\phi}}) = \frac{d}{d\tau}(2r^2\dot{\phi}) \Rightarrow \boxed{r^2\dot{\phi} = |\vec{r} \times \vec{p}| = |\vec{J}|}$.
- Variational t reads $0 = \frac{dL}{dt} - \frac{d}{d\tau}(\frac{dL}{d\dot{t}}) = 2\frac{d}{d\tau}[(1 - \frac{2GM}{r})\dot{t}] \Rightarrow \boxed{(1 - \frac{2GM}{r})\dot{t} = E}$.
- $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$ for massive objects, i.e.,
  $$1 = \frac{E^2}{1 - 2GM/r} - \frac{\dot{r}^2}{1 - 2GM/r} - \frac{J^2}{r^2} \Rightarrow \dot{r}^2 = (E^2 - 1) + \frac{2GM}{r} - \frac{J^2}{r^2} + \frac{2GMJ^2}{r^3}.$$
  Then, $\frac{\dot{r}^2 r^4}{J^2} = \left(\frac{\dot{r}r^2}{J}\right)^2 = \frac{(E^2-1)}{J^2}r^4 + \frac{2GM}{J^2}r^3 - r^2 + 2GMr.$
- $\frac{|\vec{J}|}{r^2} = \dot{\phi} = \frac{d\phi}{dr}\dot{r} \xrightarrow{\frac{dr}{d\phi} = \frac{\dot{r}r}{|\vec{J}|}} \boxed{\left(\frac{dr}{d\phi}\right)^2 = \frac{(E^2-1)}{J^2}r^4 + \frac{2GM}{J^2}r^3 - r^2 + 2GMr}$

Weak Gravity Source

Precession of Mercury's Perihelion

- $\left(\frac{d\phi}{dr}\right) = \frac{1}{\sqrt{\frac{(E^2-1)}{J^2}r^4 + \frac{2GM}{J^2}r^3 - r^2 + 2GMr}}$.
- $\phi_+ - \phi_- = \int_{R_-}^{R_+} \frac{dr}{\sqrt{\frac{(E^2-1)}{J^2}r^4 + \frac{2GM}{J^2}r^3 - r^2 + 2GMr}}$.
- The rest is math; after many steps one can solve the intergral as

$$\boxed{\phi_+ - \phi_- = \frac{\pi}{\sqrt{1-2GM/R_{||}}}\left(1+\frac{1}{4}\frac{\varepsilon}{R_{||}}\right) + \mathcal{O}\left(\frac{2GM}{R_{||}}\right)}, \text{ where}$$

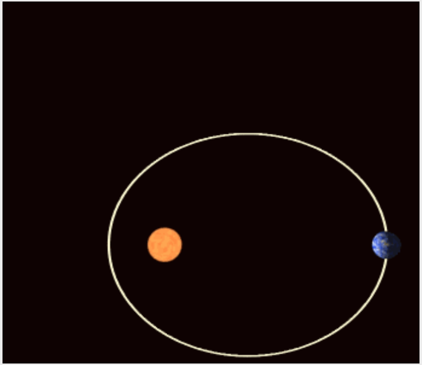
$R_{||} = \frac{R_+ R_-}{R_+ + R_-}$ and $\varepsilon = \frac{2GM}{1-2GM/R_{||}}$.

- If we use actual values, we get $\phi_+ - \phi_- = 43$ more arcseconds per century than the 531 arcseconds per century predicted by Newtonian gravity!

Weak Gravity Source

Precession of Mercury's Perihelion

Rotation of plane containing a planet elliptical trajectory



Thank You!