General Relativity Seminars

Week 5: Covariant derivative, curvature structure, and diffeomorphisms

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Outline

1. Covariant Derivatives
2. Levi-Civita Connection
3. Diffeomorphisms

Covariant Derivatives

Tensor structure

- $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu} \right)$
- $\nabla_{e_a}(e_b(s)) = e_{b,a}(s) - \Gamma^{c}_{ab} e_c(s)$
- Define ∇ as a function that takes X, Y to $\nabla_X Y$ with the following properties:
  i. $\nabla_{fX+gY} Z = f \nabla_X Z + g \nabla_Y Z$, for f, g smooth functions.
  ii. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$.
  iii. $\nabla_X (fY) = f \nabla_X Y + Y \nabla_X (f) = f \nabla_X Y + Y X(f) = f \nabla_X Y + Y df(X)$.
- Property iii. prevents defining ∇_X as a tensor since it is not linear.
  However, property i. allows us to define a $(1,1)$ tensor as follows:
  I. $\nabla Y : T_p\mathcal{M} \to T_p\mathcal{M}$ s.t. $\nabla Y : X \to \nabla_X Y$.
  II. $\nabla Y(\xi, X) : T^*_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ s.t. $\nabla Y(\xi, X) = \xi(\nabla_X Y \in \mathbb{R})$ for $\xi \in T^*_p\mathcal{M}$.

Connection components

- Define $\nabla_{e_a} e_b = \Gamma^c_{ab} e_c$, this is not the same as $\nabla_{e_a} e_b(s)$.
- $\nabla_X Y = \nabla_{X^a e_a}(Y^b e_b) = X^a \nabla_{e_a}(Y^b e_b) = X^a e_b \nabla_{e_a} Y^b + X^a Y^b \nabla_{e_a} e_b$

  $= e_b \nabla_X Y^b + X^a Y^b \Gamma^c_{ab} e_c = e_c \nabla_X Y^c + X^a Y^b \Gamma^c_{ab} e_c$

  $= X^a (\partial_{e_a} Y^c + Y^b \Gamma^c_{ab}) e_c$

  Thus for $\{\partial_\mu\}$, we have $\nabla_X Y^\gamma = X^\alpha (\partial_\alpha Y^\gamma + Y^\beta \Gamma^\gamma_{\alpha\beta})$
- In comparison with $\nabla_{e_a}(e_b(s)) = e_{b,a}(s) - \Gamma^c_{ab} e_c(s)$ there is a negative sign. This is because Y^γ is the component of a contravariant vector while $e_b(s)$ can be treated as the covariant component of a covariant vector.
- This motivates defining $\nabla_{e_a} f^b = -\Gamma^b_{ac} f^c$.
- Also, $\nabla_\alpha \xi_\beta = \partial_\alpha \xi_\beta - \Gamma^\gamma_{\alpha\beta} \xi_\gamma$ for covariant component ξ_β of a covariant vector ξ.
- To save ink and paper, replace $\{\nabla_\alpha\} \to \{;\alpha\}$ same as replacing $\{\partial_\alpha\} \to \{,\alpha\}$.

Covariant Derivatives

Other properties

- $$\boxed{T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s;\alpha} = T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s,\alpha} + \Gamma^{\mu_1}_{\alpha\beta} T^{\beta\cdots\mu_r}{}_{\nu_1\cdots\nu_s} + \cdots - \Gamma^{\beta}_{\alpha\nu_1} T^{\mu_1\cdots\mu_r}{}_{\beta\cdots\nu_s} - \cdots}$$
- $\nabla_Y \nabla_X(f) = \nabla_Y(X(f)) \xRightarrow{\text{coordinate basis}} f_{;\mu\nu} = (f_{,\mu})_{;\nu} = f_{,\mu\nu} - \Gamma^{\lambda}_{\mu\nu} f_{,\lambda}$
- Since $f_{,[\mu\nu]} = 0$ set a condition s.t. $f_{;[\mu\nu]} = 0$ which is $\Gamma^{\alpha}_{[\mu\nu]} = 0$. [coordinate basis too]
  If not, then define $2\Gamma^{\alpha}_{[\mu\nu]} := T^{\alpha}_{\mu\nu}$ which is a tensor called "torsion"!
  In presence of torsion $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu} - T_{\nu\alpha\beta} - T_{\nu\beta\alpha} + T_{\alpha\beta\nu}\right)$.
- All tensor treatments in this course assume torsion-free construction.
- $Y(X(f))$ is not a tensor since it is not linear.
  But one can prove that $\left[X, Y\right](f)$ is indeed a tensor. [in any basis]
- $\nabla_X X = X^{\nu}(X^{\mu}{}_{,\nu} + \Gamma^{\mu}_{\nu\alpha} X^{\alpha}) e_{\mu} \xRightarrow{\text{coordinate basis}} \ddot{x}^{\mu} + \Gamma^{\mu}_{\nu\alpha}\dot{x}^{\alpha}\dot{x}^{\nu} = 0$
  This is another definition of geodesic $\boxed{\nabla_X X = 0}$

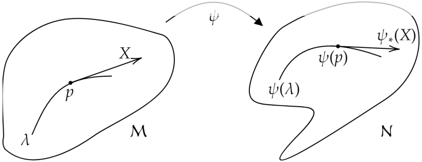
Metricity

- If $\nabla_{e_c} g_{ab} = 0$ then ∇ is called Levi-Civita connection.
  It means that the inner product defined by the metric is conserved even if the components are transferred along the curve that connection basis is tangent to, i.e., $\nabla_X g(X,X) = 0$. Componentwise and wlog this means that
  $g_{\alpha\beta;\mu} = g_{\alpha\beta,\mu} - \Gamma^{\nu}_{\mu\alpha} g_{\nu\beta} - \Gamma^{\nu}_{\mu\beta} g_{\alpha\nu} = 0 \xrightarrow{\text{LC}\nabla} g_{\alpha\beta,\mu} = \Gamma^{\nu}_{\mu\alpha} g_{\nu\beta} + \Gamma^{\nu}_{\mu\beta} g_{\alpha\nu}.$
- A determinant $g = \hat{g}^{\alpha\beta}\Delta_{\alpha\beta}$, $\Delta_{\alpha\beta} = g\hat{g}_{\alpha\beta}$ is the cofactor, defines $\dfrac{\partial g}{\partial \hat{g}^{\alpha\beta}} = \Delta_{\alpha\beta}$.
  Also, $\partial_\mu g = \dfrac{\partial g}{\partial \hat{g}^{\alpha\beta}} \partial_\mu \hat{g}^{\alpha\beta} = \Delta_{\alpha\beta}\partial_\mu \hat{g}^{\alpha\beta} = g\hat{g}_{\alpha\beta}\partial_\mu \hat{g}^{\alpha\beta} = g\hat{g}^{\alpha\beta}(\Gamma^{\nu}_{\mu\alpha}\hat{g}_{\nu\beta} + \Gamma^{\nu}_{\mu\beta}\hat{g}_{\alpha\nu}).$

$$\boxed{\therefore\ \partial_\mu g = 2g\Gamma^{\alpha}_{\alpha\mu}}$$

Diffeomorphisms

Pull-back & Push-forward



- For m-dim $\mathcal{M}$ and n-dim $\mathcal{N}$ manifolds, a function $\psi : \mathcal{M} \to \mathcal{N}$ is smooth $\iff \phi_{\mathcal{N}} \circ \psi \circ \phi^{-1}_{\mathcal{M}}$ is smooth.
- For $f : \mathcal{N} \to \mathbb{R}$, a pull-back is $\psi^*(f) : \mathcal{M} \to \mathbb{R}$ s.t. $\psi^*(f)(p) = f(\psi^*(p))$, for $p \in \mathcal{M}$.
- For $\lambda : \mathbb{R} \to \mathcal{M}$ a push-forward is $\psi_* : T_p\mathcal{M} \to T_{\psi(p)}\mathcal{N}$ s.t. $\psi_*(X) \in T_{\psi(p)}\mathcal{N}$ for $X \in T_p\mathcal{M}$.
- $(\psi_* X)f = \frac{d}{dt}(f \circ (\psi \circ \lambda(t))) = \frac{d}{dt}((f \circ \psi) \circ \lambda(t)) = X(\psi^*(f)) \Rightarrow \boxed{(\psi_* X)f = X(\psi^*(f))}$.
- For $\xi \in T^*_{\psi(p)}\mathcal{N}$ we have $\boxed{(\psi^*(\xi))(X) = \xi(\psi_*(X))}$ where $X \in T_p\mathcal{M}$.
- If $\xi \equiv df$ then $(\psi^*(df))(X) = df(\psi_*(X)) = (\psi_*(X))(f) = X(\psi^*(f)) = d(\psi^*(f))(X)$, i.e., the operator d and the morphism ψ^* are commutative $\boxed{(\psi^*(df))(X) = d(\psi^*(f))(X)}$.
- For coordinate bases $\{x^\alpha\}$ of $\mathcal{M}$ and $\{y^\mu\}$ of $\mathcal{N}$, the pull-back/push-forward is defined as

$$(\psi^*(f))(X)\Big|_p = X(\psi_*(f \circ \psi(\lambda)))\Big|_p = \frac{d(\psi_* f)}{dt} = \frac{\partial \psi_* f}{\partial \psi(\phi_{\mathcal{M}})} \frac{d\psi(\phi_{\mathcal{M}})}{dt} = \frac{\partial \psi_* f}{\partial y^\mu} \frac{dy^\mu}{dt} =$$

$$\frac{\partial \psi_* f}{\partial y^\mu} \frac{\partial y^\mu}{\partial x^\alpha} \frac{dx^\alpha}{dt} \xRightarrow{\text{coordinate bases}} \boxed{\frac{\partial y^\mu}{\partial x^\alpha} X^\alpha \longleftrightarrow \frac{\partial x^\alpha}{\partial y^\mu} \psi_*(X^\mu)} \text{ or } \boxed{\frac{\partial y^\mu}{\partial x^\alpha} \psi^*(\xi_\mu) \longleftrightarrow \frac{\partial x^\alpha}{\partial y^\mu} \xi_\alpha}$$

Isometry

- $(\psi^* S)_{\alpha_1 \cdots \alpha_s} = (\frac{\partial y^{\mu_1}}{\partial x^{\alpha_1}}) \cdots (\frac{\partial y^{\mu_s}}{\partial x^{\alpha_s}}) S_{\mu_1 \cdots \mu_s}$

  $(\psi_* S)^{\mu_1 \cdots \mu_r} = (\frac{\partial y^{\mu_1}}{\partial x^{\alpha_1}}) \cdots (\frac{\partial y^{\mu_r}}{\partial x^{\alpha_r}}) S^{\alpha_1 \cdots \alpha_r}$
- For $\psi : \mathcal{M} \to \mathcal{M}$, $\psi^* = (\psi_*)^{-1}$. If $\psi(T) = T$ for every $p \in \mathcal{M}$, then ψ is called "symmetry transformation of T". And if $T \equiv g(X, X)$, then ψ is "isometry".
- Best choice for ψ is that function which transfers $p \in \mathcal{M}$ to $\psi(p) \in \mathcal{M}$ along a curve $\lambda(t)$. Such morphism has $\psi_{t_1} \circ \psi_{t_2} \equiv \psi_{t_1 + t_2}$. Also it means $\psi_{-t} \equiv (\psi_t)^{-1}$.
- This previous property satisfies the definition of "homomorphism". What about the "diffeo" part?

Diffeomorphisms

Lie derivatives

- For a manifold $\mathcal{N}$ with $\tilde{\nabla}$ and another manifold $\mathcal{M}$ with ∇ we define $\boxed{\tilde{\nabla}_X T := \psi_* \left(\nabla_{\psi^*(Y)}(\psi^*(T))\right)}$, where Y, T are in $\mathcal{N}$.
- This is not enough since we saw before that ∇_X is not linear. We need to develop another kind of derivatives that are linear when acting on vectors.
- $\boxed{\mathcal{L}_X T := \lim_{t \to 0} \frac{((\psi_{-t})_* T) - T}{t}\bigg|_p}$
- $\mathcal{L}_X(aS + bT) = a\mathcal{L}_X(S) + b\mathcal{L}_X(T)$. And $\mathcal{L}_X(S \otimes T) = \mathcal{L}_X(S) \otimes T + S \otimes \mathcal{L}_X(T)$.
- $\mathcal{L}_X f = X(f) = df(X)$.
- $\mathcal{L}_X Y = [X, Y] = (X^\nu Y^\mu_{,\nu} - Y^\nu X^\mu_{,\nu}) e_\mu \xrightarrow{\text{prove}} (\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X) T = \mathcal{L}_{[X,Y]} T$.
- $(\mathcal{L}_X \xi)_\mu Y^\mu = \mathcal{L}_X(\xi_\mu Y^\mu) - \xi_\mu (\mathcal{L}_X Y)^\mu =$ $\xi_\mu X^\nu Y^\mu{}_{,\nu} + Y^\mu X^\nu \xi_{\mu,\nu} - \xi_\mu X^\nu Y^\mu{}_{,\nu} + \underbrace{\xi_\mu Y^\nu X^\mu{}_{,\nu}}_{\mu \longleftrightarrow \nu} \Rightarrow \boxed{(\mathcal{L}_X \xi)_\mu = X^\nu \xi_{\mu,\nu} + \xi_\nu X^\nu{}_{,\mu}}$

Killing Vectors

- $\mathcal{L}_X g = \mathcal{L}_X(g_{\mu\nu}X^\mu X^\nu) = X^\alpha g_{\mu\nu,\alpha}X^\mu X^\nu + g_{\mu\nu}X^\alpha X^\mu{}_{,\alpha}X^\nu + g_{\mu\nu}X^\alpha X^\nu{}_{,\alpha}X^\mu$
$$= X^\alpha g_{\mu\nu,\alpha}X^\mu X^\nu + \underbrace{X^\alpha X^\mu{}_{,\alpha}X_\mu}_{\alpha\longleftrightarrow\nu} + \underbrace{X^\alpha X^\nu{}_{,\alpha}X_\nu}_{\alpha\longleftrightarrow\mu}$$
$$= X^\alpha g_{\mu\nu,\alpha}X^\mu X^\nu + X^\nu X^\mu{}_{,\nu}X_\mu + X^\mu X^\nu{}_{,\mu}X_\nu$$
$$= X^\alpha g_{\mu\nu,\alpha}X^\mu X^\nu + X_{\mu,\nu}X^\mu X^\nu + X_{\nu,\mu}X^\mu X^\nu$$
$$\boxed{\therefore (\mathcal{L}_X g)_{\mu\nu} = X^\alpha g_{\mu\nu,\alpha} + \xi_{\mu,\nu} + \xi_{\nu,\mu}}$$
- The basis-independent form of the last result is $(\mathcal{L}_X g)_{mn} = X^a g_{mn;a} + \xi_{m;n} + \xi_{n;m}$
- The 1st term dies because of metricity. Therefore for an isometry $\mathcal{L}_x g = 0$, $\boxed{\nabla_{(a}\xi_{b)} = 0}$ where ξ_a is the Killing field.
- We can obtain same equation if we consider variation w.r.t $y^\mu = x^\mu + \varepsilon X^\mu$.
- Killing vectors help defining "conserved currents" $J^a = T^{ab}\xi_b$ as $\boxed{\nabla_a J^a = 0}$.

Thank You!