General Relativity Seminars

Week 4: Christoffel symbols & the extrinsic curvature tensor

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Outline

1. Geodesics
2. Christoffel Symbols
3. Extrinsic Curvature Tensor
4. Covariant Derivative

Geodesic Equation from Calculus of Variation

- $f(x(t,\varepsilon)) = f(x(t)) + \varepsilon\xi$, where ξ captures the tensorial properties of $f(x)$. If $f := x^\mu \Rightarrow \frac{\delta x^\mu}{\delta\varepsilon} = \xi^\mu$. Also, $\frac{d}{dt}\left(\frac{\delta x^\mu}{\delta\varepsilon}\right) = \frac{\delta}{\delta\varepsilon}\left(\frac{dx^\mu}{dt}\right) = \frac{\delta\dot{x}^\mu}{\delta\varepsilon} = \dot{\xi}^\mu$
- $\ell = \int dtL \xRightarrow{\delta\ell=0} \frac{\delta L}{\delta\varepsilon} = \frac{\partial L}{\partial x^\mu}\frac{\delta x^\mu}{\delta\varepsilon} + \frac{\partial L}{\partial\dot{x}^\mu}\frac{\delta\dot{x}^\mu}{\delta\varepsilon} = \boxed{\frac{\partial L}{\partial x^\mu}\xi^\mu + \frac{\partial L}{\partial\dot{x}^\mu}\dot{\xi}^\mu = 0}$
- Let $\ell = \int dt\sqrt{-g_{ab}U^aU^b} = \int dt\sqrt{-g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}$
- $\frac{dL}{dx^\mu} = -\frac{1}{2L}\partial_\mu(g_{\alpha\beta})\dot{x}^\alpha\dot{x}^\beta$
- $\frac{\partial L}{\partial\dot{x}^\mu} = -\frac{1}{2L}g_{\alpha\beta}\frac{\partial\dot{x}^\alpha}{\partial\dot{x}^\mu}\dot{x}^\beta = -\frac{1}{2L}g_{\alpha\beta}\delta^\alpha{}_\mu\dot{x}^\beta = -\frac{1}{2L}g_{\mu\beta}\dot{x}^\beta$

  Similarly $\frac{\partial L}{\partial\dot{x}^\mu} = -\frac{1}{2L}g_{\alpha\mu}\dot{x}^\alpha = -\frac{1}{2L}g_{\mu\alpha}\dot{x}^\alpha$

  $\Bigg\} \frac{\partial L}{\partial\dot{x}^\mu} = -\frac{1}{2L}2g_{\mu\alpha}\dot{x}^\alpha$

Geodesic Equation from Calculus of Variation

- $$\frac{\partial L}{\partial x^\mu}\xi^\mu + \frac{\partial L}{\partial \dot{x}^\mu}\dot{\xi}^\mu = -\frac{1}{2L}\left[\partial_\mu(g_{\alpha\beta})\dot{x}^\alpha\dot{x}^\beta\xi^\mu + 2g_{\mu\alpha}\dot{x}^\alpha\dot{\xi}^\mu\right]$$
$$= -\frac{1}{2L}\left[\partial_\mu(g_{\alpha\beta})\dot{x}^\alpha\dot{x}^\beta\xi^\mu + \cancel{2\frac{d}{dt}(g_{\mu\alpha}\dot{x}^\alpha\xi^\mu)} - 2g_{\mu\alpha}\ddot{x}^\alpha\xi^\mu - 2\frac{d}{dt}(g_{\mu\alpha})\dot{x}^\alpha\xi^\mu\right]$$
$$= -\frac{1}{2L}\left[\partial_\mu(g_{\alpha\beta})\dot{x}^\alpha\dot{x}^\beta - 2g_{\mu\alpha}\ddot{x}^\alpha - 2\partial_\beta(g_{\mu\alpha})\dot{x}^\alpha\dot{x}^\beta\right]\xi^\mu = 0$$
- $\therefore\ 2g_{\mu\alpha}\ddot{x}^\alpha + 2\partial_\beta(g_{\mu\alpha})\dot{x}^\alpha\dot{x}^\beta - \partial_\mu(g_{\alpha\beta})\dot{x}^\alpha\dot{x}^\beta = 0$

  Or $\quad g_{\mu\alpha}\ddot{x}^\alpha + \partial_\beta(g_{\mu\alpha})\dot{x}^\alpha\dot{x}^\beta - \frac{1}{2}\partial_\mu(g_{\alpha\beta})\dot{x}^\alpha\dot{x}^\beta = 0$

  Or $\quad g_{\mu\alpha}\ddot{x}^\alpha + \frac{1}{2}\partial_\beta(g_{\mu\alpha})\dot{x}^\alpha\dot{x}^\beta + \underbrace{\frac{1}{2}\partial_\beta(g_{\alpha\mu})\dot{x}^\alpha\dot{x}^\beta}_{\text{dummy }\alpha\longleftrightarrow\beta} - \frac{1}{2}\partial_\mu(g_{\alpha\beta})\dot{x}^\alpha\dot{x}^\beta = 0$

$$g_{\mu\alpha}\ddot{x}^\alpha + \frac{1}{2}\left[\partial_\beta(g_{\mu\alpha})\dot{x}^\alpha\dot{x}^\beta + \partial_\alpha(g_{\underline{\beta\mu}})\underline{\dot{x}^\beta\dot{x}^\alpha} - \partial_\mu(g_{\alpha\beta})\dot{x}^\alpha\dot{x}^\beta\right] = 0$$

Geodesic Equation from Calculus of Variation

- $$g_{\mu\alpha}\ddot{x}^\alpha + \frac{1}{2}\Big[\partial_\beta(g_{\mu\alpha})\dot{x}^\alpha\dot{x}^\beta + \partial_\alpha(g_{\mu\beta})\dot{x}^\alpha\dot{x}^\beta - \partial_\mu(g_{\alpha\beta})\dot{x}^\alpha\dot{x}^\beta\Big] = 0$$
  Or $$g_{\mu\alpha}\ddot{x}^\alpha + \frac{1}{2}\Big[\partial_\beta(g_{\mu\alpha}) + \partial_\alpha(g_{\mu\beta}) - \partial_\mu(g_{\alpha\beta})\Big]\dot{x}^\alpha\dot{x}^\beta = 0$$
- $$\boxed{\Gamma_{\mu\alpha\beta} := \frac{1}{2}\Big[\partial_\beta(g_{\mu\alpha}) + \partial_\alpha(g_{\mu\beta}) - \partial_\mu(g_{\alpha\beta})\Big]}$$, which is the "Christoffel symbol of 1st kind".
- We can save ink and paper if $\partial_\alpha f \to f_{,\alpha}$ or $e_a(f) \to f_{,a}$
- $$\boxed{\therefore \Gamma_{\mu\alpha\beta} = \frac{1}{2}\Big[(g_{\mu\alpha,\beta}) + (g_{\mu\beta,\alpha}) - (g_{\alpha\beta,\mu})\Big]}$$
- $$\boxed{\Gamma^\mu_{\alpha\beta} := \frac{1}{2}g^{\mu\nu}\Big[(g_{\nu\alpha,\beta}) + (g_{\nu\beta,\alpha}) - (g_{\alpha\beta,\nu})\Big]}$$, which is the "Christoffel symbol of 2nd kind".
- Is the Christoffel symbol a tensor? NO! Why?
  $$\Gamma^\lambda_{\mu\nu} \sim \partial^\lambda g_{\mu\nu} = \mho^\lambda{}_\alpha\partial^\alpha(\Omega^\beta{}_\mu\Omega^\gamma{}_\nu g_{\beta\gamma}) \neq (\mho^\lambda{}_\alpha\Omega^\beta{}_\mu\Omega^\gamma{}_\nu)[\partial^\alpha g_{\beta\gamma}]$$

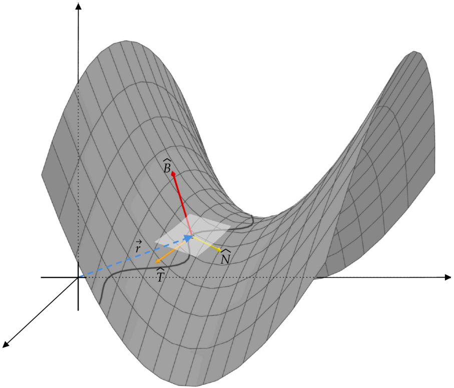
Christoffel Symbols

TNB frame & Frenet-Serret Equations

- $ds = \left|\vec{r}(t+dt) - \vec{r}(t)\right| = \sqrt{-g_{ab}\ dh^a\ dh^b}$
- $\dfrac{ds}{dt} = \dfrac{dh^a}{dt}\dfrac{\partial s}{\partial h^a} = H^a e_a(s) = H(s)$
- $\hat{T} := \dfrac{ds/dt}{|ds/dt|}$
- $\hat{N} := \dfrac{d\hat{T}/dt}{|d\hat{T}/dt|} = \dfrac{d\hat{T}/dt}{\kappa}$, κ is "curvature scalar".
- $\hat{B} := \hat{T} \times \hat{N}$
- $\dfrac{d\hat{T}}{dt} = \kappa\hat{N} \Rightarrow \dfrac{d\hat{N}}{dt} = -\kappa\hat{T}$, κ is constant per point.

$$\frac{d\hat{N}}{dt} = \langle\frac{d\hat{N}}{dt},\hat{T}\rangle\hat{T} + \cancel{\langle\frac{d\hat{N}}{dt},\hat{N}\rangle\hat{N}} + \langle\frac{d\hat{N}}{dt},\hat{B}\rangle\hat{B} = -\kappa\hat{T} + \tau\hat{B}$$

$$\frac{d\hat{B}}{dt} = \cancel{\langle\frac{d\hat{B}}{dt},\hat{T}\rangle\hat{T}} + \langle\frac{d\hat{B}}{dt},\hat{N}\rangle\hat{N} + \cancel{\langle\frac{d\hat{B}}{dt},\hat{B}\rangle\hat{B}} = -\tau\hat{N}$$, τ is "torsion", $\tau = 0$ in planar curves.



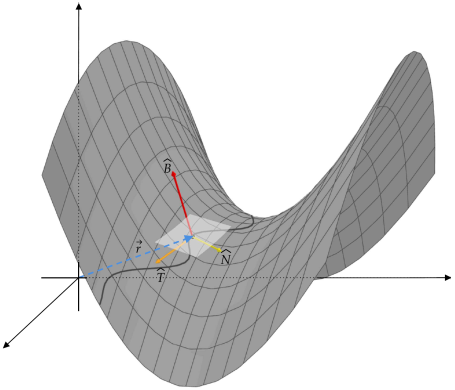
See W. Kühnel, "Differential Geometry: Curves–Surfaces–Manifolds", 3rd Ed., AMS, 2013, ISBN: 9781470423209, §1-2.

Christoffel Symbols

Basis-Independent Geodesic Equation

- $$\frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d}{dt}\left(H^a e_a(s)\right) = H^b e_b\left(H^a e_a(s)\right)$$
  $$= H^b e_b(H^a) e_a(s) + H^a H^b e_b(e_a(s))$$
  $$= \frac{d}{dt}\left(\frac{dh^a}{dt}\right)\frac{\partial s}{\partial h^a} + \frac{dh^a}{dt}\frac{dh^b}{dt}\frac{\partial^2 s}{\partial h^b \partial h^a}$$
- $$\left.\frac{d^2 x^\mu}{dt^2}\right|_{\text{static}} = \frac{d^2 x^\alpha}{dt^2}\frac{\partial x^\mu}{\partial x^\alpha} + \frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\frac{\partial^2 x^\mu}{\partial x^\beta \partial x^\alpha}$$
  $$= \frac{d^2 x^\alpha}{dt^2}\delta^\mu{}_\alpha + \frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\frac{\partial^2 x^\mu}{\partial x^\beta \partial x^\alpha}$$
  $$\boxed{\left.\frac{d^2 x^\mu}{dt^2}\right|_{\text{static}} = \left.\frac{d^2 x^\mu}{dt^2}\right|_{\text{comov.}} + u^\alpha u^\beta \frac{\partial^2 x^\mu}{\partial x^\beta \partial x^\alpha}}$$

  This is the general geometric definition of Newton's 2nd Law in rotating frames including the Euler term $\dot{\vec{w}} \times \vec{r}$, centrifugal term $\vec{w} \times (\vec{w} \times \vec{r})$, and Coriolis term $2(\vec{w} \times \dot{\vec{r}})$.
- Remember that $u^\alpha u^\beta \frac{\partial^2 s}{\partial x^\beta \partial x^\alpha} = \frac{d^2 s}{dt^2} = \frac{d\hat{T}}{dt} = \kappa \hat{N} \xrightarrow{\cdot \hat{N}} \boxed{u^\alpha u^\beta \hat{N} \cdot \partial_\beta \partial_\alpha (s) = \pm\kappa}$.

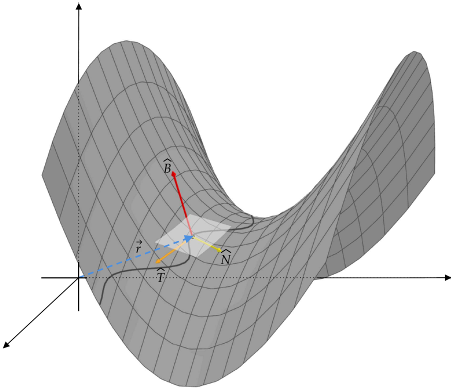


Extrinsic Curvature Tensor

Weingarten Equation

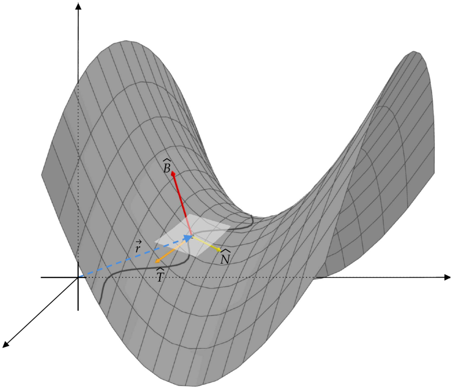
- The last result can be generalized to define the symmetric "extrinsic curvature tensor":
  $$\boxed{\mathcal{K}_{\alpha\beta}(s) := \pm\hat{N}\cdot\partial_\alpha\partial_\beta s}$$
- The former definition can be basis-independent:
  $$\boxed{\mathcal{K}_{ab}(s) := \pm\hat{N}\cdot e_a(e_b(s)) = \pm\hat{N}\cdot e_{b,a}(s)}$$
  We drop (s) to be restored later.
- Since $e_a\cdot\hat{N}=0$, then $\mathcal{K}_{ab} = \mp e_b e_a(\hat{N}) = \mp e_b\hat{N}_{,a}$.
  The sign $\pm$ is based on if $\hat{N}$ is spacelike or timelike.
- $$f^b\mathcal{K}_{ab} = \mp f^b e_b\hat{N}_{,a} \xrightarrow[\text{absorb } d \text{ in } \hat{N}]{\text{lower } f^b} g^{cb}e_c\mathcal{K}_{ab} = e_c\mathcal{K}_a{}^c = \mp\hat{N}_{,a}.$$

$$\boxed{\therefore \hat{N}_{,a} = \mp e_b\mathcal{K}_a{}^b}$$ [to be used in the section of initial value problem]



Covariant Derivative

Gauss Equation



- The $\frac{d}{dt}\hat{N} = -\kappa\hat{T} + \tau\hat{B}$ says that it has a component in the direction of $e_a(s)$. Moreover, the $\tau\hat{T}\times\hat{N}$ indicates that the components are for 3-index "object". Since κ is promoted to $\mathcal{K}_{ab}$, that is a function $e_{a,b}(s)$, one can suggest writing $\boxed{e_{a,b}(s) = \mathfrak{C}^c_{ab}e_c(s) + \mathcal{K}_{ab}(s)\hat{N}}$
- $\therefore\ e_{a,b}(s)f^c(s) = \mathfrak{C}^c_{ab}e_c f^c + \cancel{\mathcal{K}_{ab}\hat{N}f^c} = \mathfrak{C}^c_{ab}g_c{}^c = \mathfrak{C}^c_{ab}$.
  Or: $e_{a,b}e_c = (e_a e_c)_{,b} = g_{ac,b} = \mathfrak{C}_{abc}$
- With some "abuse of notation", one can derive:

  i. $g_{ac,b} = \mathfrak{C}_{abc} + \mathfrak{C}_{cba}$
  ii. $g_{cb,a} = \mathfrak{C}_{cab} + \mathfrak{C}_{bac}$
  iii. $g_{ba,c} = \mathfrak{C}_{bca} + \mathfrak{C}_{acb}$

  $$\overset{\text{i+ii−iii}}{\Longrightarrow}\ \boxed{\mathfrak{C}_{abc} = \frac{1}{2}\left[g_{ab,c} + g_{ac,b} - g_{bc,a}\right] \equiv \Gamma_{abc}}$$
- $\boxed{\therefore e_{a,b}(s) = \Gamma^c_{ab}e_c(s) + \mathcal{K}_{ab}(s)\hat{N}} \xrightarrow{\mathcal{K}_{ab}(s)\hat{N}\equiv\nabla_{e_a}e_b} \boxed{\nabla_{e_a}e_b(s) := e_{a,b}(s) - \Gamma^c_{ab}e_c(s)}$

Thank You!