General Relativity Seminars

Week 8: Black Holes & Wormholes

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Outline

1. Singularities in the Metric!
2. Event Horizon
3. Wormholes

Singularities in the Metric!

Laplace-Michell "Dark Stars"

- $\frac{1}{2}\cancel{m}v^2 = mgR = \cancel{m}\frac{GM}{R}$.
- If $v = c \Rightarrow R = \frac{2GM}{c^2}$, i.e., light cannot escape the star if its radius is less than $\frac{2GM}{c^2}$. Of course this analysis is incorrect (WHY?!) but the result miraculously matches with a "physical singularity" in Schwarzschild metric $ds^2 = -(1 - \frac{2GM/c^2}{r})dt^2 + \frac{1}{(1-\frac{2GM/c^2}{r})}dr^2 + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2$.
- This corresponds to destroying the metric by setting $r = 2GM/c^2 \to g_{00} = 0$ and $g_{11} \to \infty$. Let's choose $c = 1$, then $R_s = 2GM$ is Schwarzschild radius.

Interior Solution & Tolman–Oppenheimer–Volkoff Equation

- Previously we considered vacuum solution $T_{\mu\nu} = 0$ of Schwarzschild metric to obtain the mercury precession.
- Now we consider $T_{\mu\nu} = (\rho(r) + p(r))\, V_\mu V_\nu + p(r) g_{\mu\nu}$, where ρ is mass density, p is the pressure, and $V_\mu = [V_0, \quad \vec{0}]$ is the speed in rest frame.
- Since $V_\mu V^\mu = -1 = g^{\mu\nu} V_\mu V_\nu$, then $T_{\mu\nu} = \text{diag}(-g_{00}\rho,\ g_{11}p,\ g_{22}p,\ g_{33}p)$.
- Also, as $\rho \equiv \rho(r)$, then $1 - \dfrac{2GM}{r} \equiv 1 - \dfrac{2GM(r)}{r}$.
- Then $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, where $G_{\mu\nu} = \text{diag}(G_{00},\ G_{11},\ G_{22},\ G_{33})$.
- $G_{00} = 8\pi G T_{00} \rightarrow \dfrac{dM}{dr} = 4\pi r^2 \rho$. Knowing ρ gives solution for M.

$$G_{rr} = 8\pi G T_{rr} \xrightarrow{\nabla_r T^{rr} = 0} \boxed{\frac{dp}{dr} = -(\rho + p)\frac{GM + 4\pi G r^3 p}{r^2 - 2rGM}}.$$

Singularities in the Metric!

Buchdahl's theorem, Chandrasekhar limit, and TOV limit

- If $\rho(r) = \begin{cases} \rho_0 & , r < R \\ 0 & , r > R \end{cases} \Rightarrow M(r) = \begin{cases} \frac{4}{3}\pi\rho_0 r^3 & , r < R \\ \frac{4}{3}\pi\rho_0 R^3 & , r > R \end{cases}$.
- For $r < R$, $p(r) = -\rho_0 \dfrac{\sqrt{R^3 - 2GMr^2} - R\sqrt{R - 2GM}}{\sqrt{R^3 - 2GMr^2} - 3R\sqrt{R - 2GM}}$ [What is $p(r)$, $r > R$?]
- As $\downarrow r$, $\uparrow p(r)$. Moreover, $p(r) \xrightarrow[r\to 0]{M=4R/9G} \infty$. And if $\boxed{M > \dfrac{4R}{9G}}$, then the system is no longer time independent. The star collapses under gravity!
- Even if ρ is more complicated than ρ_0, Buchdahl's theorem says $M_{\mathrm{B}} = \frac{4R}{9G}$ is the limit, for all different forms of $\rho(r)$, after which the star collapses.
- For the radius of Sun, $M_{\mathrm{B}} \sim 1.4 \times 10^{27} kg$. The Sun mass is $M_{\odot} \sim 2 \times 10^{30} kg$!!!
- We ignored non gravitational interactions. If considered, then $M > 1.4 M_{\odot}$ gives a dwarf star limit. And if $M \sim 3 - 4 M_{\odot}$ a black hole is obtained!

Singularities in the Metric!

Coordinate Singularity vs. Physical Singularity

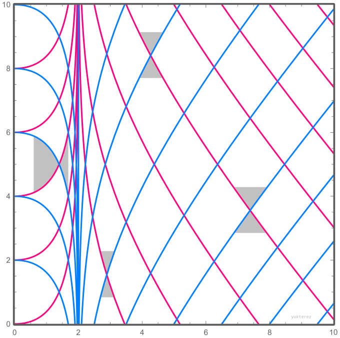
- At $r = 0$ and $r = 2GM$

$$ds^2 = -(1 - \frac{2GM/c^2}{r})dt^2 + \frac{1}{(1 - \frac{2GM/c^2}{r})}dr^2 + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2$$

  has singularities, i.e., $g_{\mu\nu}$ is degenerate.

- Physics comes from Riemann curvature. But it is NOT a coordinate invariant. Also, in vacuum, the Ricci scalar is zero. However Kretschmann scalar $R_{ijkl}R^{ijkl} = \frac{48G^2M^2}{r^2}$ which is a coordinate invariant (see Ricci decomposition).

- This tells us that $r = 0$ is the only problematic singularity as $R_{ijkl}R^{ijkl} \neq 0$ when $r = 2GM$.



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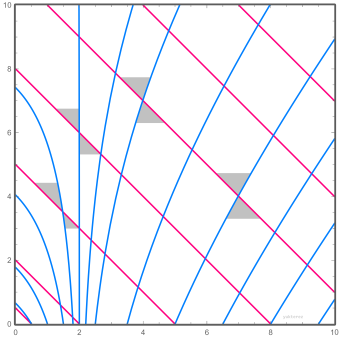
Event Horizon

Eddington–Finkelstein Coordinates

- $r^* = r + 2GM \ln|r - 2GM| \Rightarrow dr^* = \left(1 - \frac{2GM}{r}\right)^{-1} dr$
- $r^* \to \infty$ as $r = 2GM$, hence the name of tortoise coordinate after the famous Zeno of Elea's paradox.
- The ingoing $v = t + r^* \Rightarrow dt = dv - \left(1 - \frac{2GM}{r}\right)^{-1} dr$.
- $ds^2_{\text{ingo.}} = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2\,dv\,dr + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2$.
- The outgoing $u = t - r^* \Rightarrow dt = du + \left(1 - \frac{2GM}{r}\right)^{-1} dr$
- $ds^2_{\text{outgo.}} = -\left(1 - \frac{2GM}{r}\right) du^2 - 2\,du\,dr + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2$
- If $t' = t \pm (r^* - r)$,
  then $ds^2 = -dt'^2 + dr^2 + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2 + \frac{2GM}{r}(dt' \pm dr)^2 \xrightarrow{r\to\infty}$?



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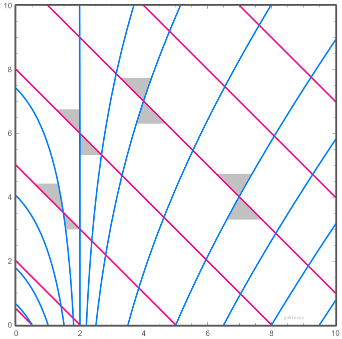
Event Horizon

Eddington–Finkelstein Coordinates

- For lightcone, $ds^2_{\text{Schw.}} = 0 \Rightarrow \dfrac{dr}{dt} = \pm\left(1 - \dfrac{2GM}{r}\right)$.

  At $r = 0$ we have $c = \left.\dfrac{dr}{dt}\right|_{\text{Hor.}} = 0$!!!
- $ds^2_{\text{EF}} = 0 \Rightarrow \begin{cases} dv = 0, \; v = \text{const.} \\ dr = 0, \; r = 2GM \\ \frac{dr}{dv} = \frac{1}{2}(1 - \frac{2GM}{r}) \end{cases}$
- For $v = \text{const.}$, it stays 45°.
- For $r = 2GM$, time t' freezes!
- For $\dfrac{dr}{dt} > 0$ has $\eta < 45^\circ$ when close to $r = 2GM$.
- For $\dfrac{dr}{dt} = 0$ has $\eta = 0^\circ$ at $r = 2GM$.
- For $\dfrac{dr}{dt} < 0$ has $\eta < 0^\circ$ when $r < 2GM$.



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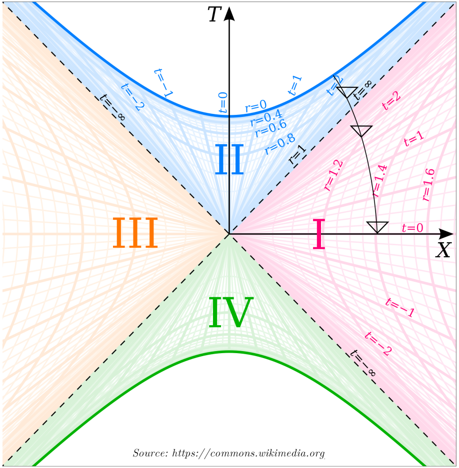
Kruskal-Szekeres Coordinates

- $T_I = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right).$
  $X_I = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right).$
- $T^2 - X^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM}.$
- $$ds^2_{\mathrm{KS}} = \frac{32G^3M^3}{r} e^{-r/2GM}(-dT^2 + dX^2) + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2$$
- $ds^2_{\mathrm{KS}} = 0 \Rightarrow dX/dT = c = 1$ always.
- $T_{II} = \left(1 - \frac{r}{2GM}\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right).$
  $X_{II} = \left(1 - \frac{r}{2GM}\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right).$
- $$\tanh\left(\frac{t}{4GM}\right) = \begin{cases} T/X & \text{(in I and III)} \\ X/T & \text{(in II and IV)} \end{cases}.$$
- $\left.\begin{array}{l} T_{III} = -T_{II} \text{ and } X_{III} = -X_{II}. \\ T_{IV} = -T_I \text{ and } X_{IV} = -X_I. \end{array}\right\}$ The spacetime is now "maximally extended".



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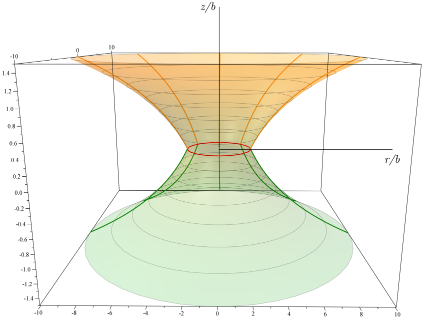
Wormholes

Einstein–Rosen Bridge

- $ds^2 = -(1 - \frac{2GM/c^2}{r})dt^2 + \frac{1}{(1-\frac{2GM/c^2}{r})}dr^2 + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2.$
- To resolve singularities of quantum particles, set $u^2 = r - 2M$.
- $ds^2_{\mathrm{ER}} = -\frac{u^2}{u^2+2M}\, dt^2 + 4(u^2+2M)\, du^2 + (u^2+2M)^2(d\theta^2 + \sin^2\theta\, d\phi^2)$
- This bridge is unstable and would collapse if it exists, proved by Robert W. Fuller and John A. Wheeler, (1962-10-15). "Causality and Multiply Connected Space-Time". Physical Review. 128 (2). American Physical Society (APS): 919–929.

Ellis Wormhole

- $ds'^2 = -dt^2 + dr'^2 + (b^2 + r'^2)(d\theta'^2 + \sin^2\theta\, d\phi'^2)$.
- To consider cylindrical symmetry, set $dt = 0$ and $\theta' = \pi/2 \Rightarrow \boxed{ds'^2 = dr'^2 + (r'^2 + b^2)d\phi^2}$.
- Embed ds'^2 in $ds^2_{\text{cyl.}} = dz^2 + dr^2 + r^2 d\phi^2$ s.t. $\boxed{ds^2_{\text{cyl.}} = (\frac{\partial z}{\partial r'})^2 dr'^2 + (\frac{\partial r}{\partial r'})^2 dr'^2 + r^2 d\phi^2}$.
- By comparison between ds'^2 and $ds^2_{\text{cyl.}}$, $r^2 = r'^2 + b^2$ and $\partial r/\partial r' = r'/\sqrt{r'^2 + b^2}$.
- Cylindrical symmetry demands that $(\frac{\partial z}{\partial r'})^2 + (\frac{\partial r}{\partial r'})^2 = 1$. Then $(\frac{\partial z}{\partial r'})^2 + (\frac{r'}{\sqrt{r'^2 + b^2}})^2 = 1 \Rightarrow \boxed{z = b \sinh^{-1}(\frac{r'}{b}) = b \sinh^{-1}(\sqrt{\frac{r^2}{b^2} - 1})}$.
- Double check that $ds^2_{\text{cyl.}} \to ds'^2 = dr'^2 + (r'^2 + b^2)d\phi^2$



Thank You!