General Relativity Seminars

Week 3: Manifolds & tensor structures

Hassan Alshal

Outline

1. Manifolds & Charts
2. Tangent & Cotangent Vectors
3. Tensors
4. Metric Tensor
5. Curve Length

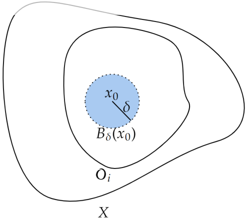
Is that you again?!

A *manifold* is a topological space that looks (i.e. it is homeomorphic to) locally (i.e. in a *patch*) like a piece of $\mathbb{R}^d$. d is the dimension of the manifold and the correspondence between the patch and the piece of $\mathbb{R}^n$ can be used to label the points in the patch by Cartesian $\mathbb{R}^n$ coordinates x^μ. In the overlap between different patches the different coordinates are consistently related by a *general coordinate transformation* (GCT) $x'^\mu(x)$. Only objects with good transformation properties under GCTs can be defined globally on the manifold. These objects are *tensors*.

T. Ortín, Gravity and Strings, CUP, 2nd Ed. (2015), p. 3

Differential Geometry is not a geometric framework in the analytical sense. It is about doing calculus on functions living on spaces rather than on $\mathbb{R}$.

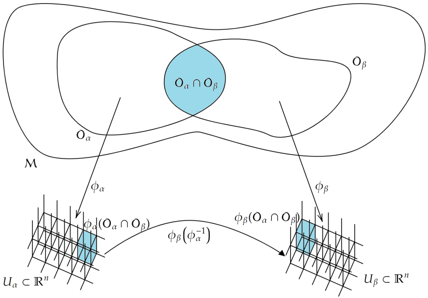
What is a topological space?



- For a set X, we can define a "distance function" $d : X \times X \to \mathbb{R}^+ \bigcup \{0\},\ d(x_0, x_i) = \delta_0$. (X, d) is called metric set
- VIP: $d(x_0, x_i) > 0$ for $x_0 \neq x_i$ to avoid pseudometric spaces.
- A "ball" $B_\delta(x_0) = \{x_i \in X | d(x_0, x_i) < \delta\}$, i.e., δ draws a boundary for x_i.
- However, B_δ has no boundary as $d(x_0, x_i) \lneq \delta$, i.e., B_d is "Open Set".
- If there is a function $f : (X, d) \to (X', d')$, the ball structure guarantees that f is continuous, i.e., $B_{\delta'}(f(x_0)) = \{f(x_i) \in X' | d'(f(x_0), f(x_i)) < \delta'\}$.
- For $\{\mathcal{O}_j | B_\delta \subset \mathcal{O}_j\}$, then $\bigcup\limits_{\text{arbitrary}} \mathcal{O}_j$ and $\bigcap\limits_{\text{finite}} \mathcal{O}_j$ are open $\Rightarrow \mathcal{O}_d(X) = \{\mathcal{O}_j\}$.
- $\mathcal{O}_d(X)$ becomes $\mathcal{T}$ if for $\mathcal{O}_j \in \mathcal{T}$, then $\varnothing, X, \bigcup\limits_{\text{arbitrary}} \mathcal{O}_j, \bigcap\limits_{\text{finite}} \mathcal{O}_j \in \mathcal{T}$.
- $(X, \mathcal{T})$ is the so called the "topological space".

What is a manifold?

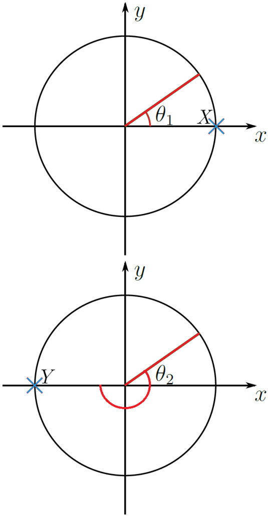
- A manifold $\mathcal{M}$ is a metric space endowed with a topological structure, as described before s.t. $\bigcup \mathcal{O}_\alpha = \mathcal{M}$, and a complete atlas.
- An atlas is $\{\phi_\alpha | \phi_\alpha : \mathcal{O}_\alpha \rightarrowtail\!\!\!\rightarrow \mathcal{U}_\alpha,\ \mathcal{U}_\alpha \subset \mathbb{R}^n\}$. This is the correspondence between patches and pieces of $\mathbb{R}^n$. To maintain smoothness, the overlap between patches demands for $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ that $\phi_\beta(\phi_\alpha^{-1}) : \phi_\alpha(\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \longrightarrow \phi_\beta(\mathcal{O}_\alpha \cap \mathcal{O}_\beta)$.
- These ϕ_α's are "charts" or "coordinate systems"!
- There are many ways to form $\{\phi_\alpha\}$. The set containing all those possible chart sets $\bigcup\{\phi_\alpha\}$ is the complete atlas.



Manifolds & Charts

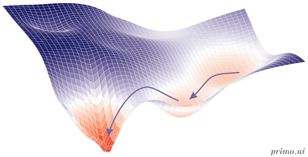
Charts & Smooth Fuctions

- After all, we can think of a chart as $\phi \equiv (x_1, x_2, \cdots, x_n)$.
- For S^1, the usual chart $\phi(\theta) := \phi(e^{i\theta}) = (\cos\theta, \sin\theta)$ cannot cover S^1 as the point X has two values. Even if you make a branch as in complex surfaces, $\phi(0) = \phi(2\pi)$.
- So, define $\phi_\alpha : S^1 \backslash \{X\} \to [-1,1] \times [-1,1] \backslash \{(1,0)\}$.
- Then, define $\phi_\beta : S^1 \backslash \{Y\} \to [-1,1] \times [-1,1] \backslash \{(-1,0)\}$.
- $\mathcal{O}_\alpha \cap \mathcal{O}_\beta = S^1 \backslash \{X, Y\}$, and $\mathcal{O}_\alpha \cup \mathcal{O}_\beta = S^1$
  $\Rightarrow \phi_\beta(\phi_\alpha^{-1})(\theta_1) = (\cos(\theta_1 - \pi), \sin(\theta_1 - \pi))$, i.e., $\theta_2 = \theta_1 - \pi$.
- A function $f : \mathcal{M} \to \mathbb{R}$ is smooth $\iff$ for any ϕ_α we have a smooth $F \equiv f(\phi_\alpha^{-1}) : \mathcal{U}_\alpha \to \mathbb{R}$. Such f is called "scalar field".



Charts & Smooth Fuctions

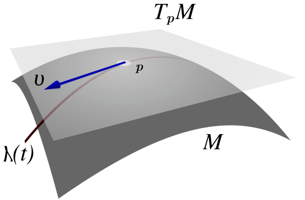
- A smooth curve in $\mathcal{M}$ is a smooth function $\lambda : I \to \mathcal{M}$, and $I \subset \mathbb{R}$.
- λ is part of $\mathcal{M}$, meanwhile f lives on $\mathcal{M}$. Thus, we need to define f according to λ in order to do calculus on f.
- So for a parameter t, we define $\frac{d}{dt}(f(\lambda)) = \frac{df}{d\lambda}\frac{d\lambda}{dt}$.
- But $\frac{df}{d\lambda}$ is ambiguous as we do calculus in terms of parameters not curves. Nevertheless, $\phi(\lambda)$ is indeed a smooth function and eventually it gives $(x_1, x_2, \cdots, x_n)$ where x_i can be treated as a parameter.



Contravariant Vectors

- $\therefore f(\lambda) \equiv f(\phi^{-1}(\phi(\lambda))) = f(\phi^{-1}) \circ (\phi(\lambda))$

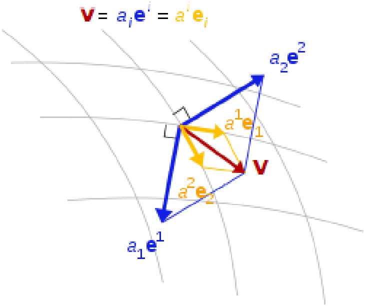
$$\frac{d}{dt}\Big(f(\phi^{-1}) \circ (\phi(\lambda))\Big) = \partial_\mu F(x^\nu) \frac{d}{dt}\Big(x^\mu(\lambda)\Big)$$
$$= (u^\mu)\partial_\mu f$$



- This takes the form $u(f) = u^\mu \hat{e}_\mu(f)$
- $\boxed{\therefore \frac{d}{dt}(f(\lambda)) \equiv u(f)}$ forms a "Tangent Space" $T_p\mathcal{M}$.
- Linear independence: $f(q) = f(p) + (x^\mu - a^\mu)\partial_\mu f\Big|_{x^\mu = a^\mu}$
- Span condition is left for you to prove!
- $\boxed{\partial_\mu = \left(\frac{\partial x^\alpha}{\partial x^\mu}\right)\partial_\alpha}$, where $\left(\frac{\partial x^\alpha}{\partial x^\mu}\right)$ is the general coordinate transformation.

Tangent & Cotangent Vectors

Covariant Vectors



- How about $df = \dfrac{\partial f}{\partial x^\mu} dx^\mu$?
- $df = \dfrac{\partial f}{\partial x^\mu} dx^\mu = \partial_\mu f \dfrac{dx^\mu}{dt} dt = \partial_\mu f u^\mu dt = \partial_\mu f u^\mu \dfrac{dt}{dx^\nu} dx^\nu$
- So $df(u^\nu) = df(\dfrac{dx^\nu}{dt}) = [\partial_\mu f u^\mu] dx^\nu$
- Just how $u \in T_p\mathcal{M}$, we have the "Cotangent Space" $T_p^*\mathcal{M} := \{df = (df)_\nu \hat{f}^\nu\}$.
- One can prove $\boxed{df(u) = u(f)}$
- and can also prove $\boxed{dx^\nu \Big(\partial_\mu\Big) = \hat{f}^\nu \hat{e}_\mu \equiv df_\nu \Big(u^\mu\Big) = \delta^\nu{}_\mu}$
- and finally prove $\boxed{dx^\nu = \Big(\dfrac{\partial x^\nu}{\partial x^\alpha}\Big) dx^\alpha}$

Tensors

A Multilinear Map

- $$T : \overbrace{T_p^*\mathcal{M} \times \cdots \times T_p^*\mathcal{M}}^{r-\text{tuple}} \times \underbrace{T_p\mathcal{M} \times \cdots \times T_p\mathcal{M}}_{s-\text{tuple}} \longrightarrow \mathbb{R} \quad , \quad \dim(T) = n^{r+s}$$ [why?]

  This means given r covariant vectors $\{\hat{f}^\mu\}$ and s contravariant vectors $\{\hat{e}_\nu\}$, then a $T(r,s)$ tensor mixes them to get a real number.
- The tensor that sends a covariant vector to real numbers is a $T(1,0)$ or the contravariant tensor as $V(f) \in \mathbb{R}$. Componentwise $T(\hat{f}^\mu) = T^\mu$.
- The tensor that sends a contravariant vector to real numbers is a $T(0,1)$ or the covariant tensor as $df(V) \in \mathbb{R}$. Componentwise $T(\hat{e}_\mu) = T_\mu$.
- $T(1,1)$ mixes both s.t. $T : T_p^*\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$. Componentwise $T(\hat{f}^\mu, \hat{e}_\nu) = T^\mu_{\ \nu}$.
- $V(f) = df(V) = T(df, V)$ but $V \in T_p\mathcal{M}$ and $df \in T_p^*\mathcal{M}$.

Tensors

Ordering & Abstract Index Notation

From now on, we drop the hat off the bases, assign X, Y, Z for contravariant vectors, and assign ζ, ξ, ω for covariant vectors.

- $\left(\dfrac{\partial x^\alpha}{\partial x^\mu}\right)$ is a special case of $\Omega^\alpha{}_\mu$. It is a basis dependent transformation when the basis $\left\{\partial_\alpha\right\}$.

- If action of a tensor or on a tensor is basis independent, replace greek indices with latin ones, i.e., $\left(\dfrac{\partial x^\alpha}{\partial x^\mu}\right) \longrightarrow \Omega^a{}_m$ and $\left\{\partial_\alpha\right\} \longrightarrow \left\{e_a\right\}$.

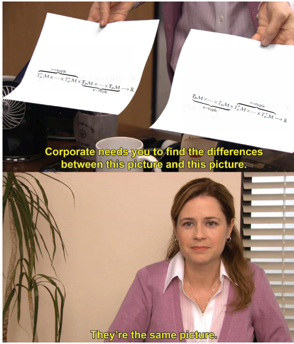
- What is $T : \overbrace{T_p^*\mathcal{M} \times \cdots \times T_p^*\mathcal{M}}^{r-\text{tuple}} \times \underbrace{T_p\mathcal{M} \times \cdots \times T_p\mathcal{M}}_{s-\text{tuple}} \longrightarrow \mathbb{R}$, $\dim(T) = n^{r+s}$

  compared to $T' : \underbrace{T_p\mathcal{M} \times \cdots \times T_p\mathcal{M}}_{s-\text{tuple}} \times \overbrace{T_p^*\mathcal{M} \times \cdots \times T_p^*\mathcal{M}}^{r-\text{tuple}} \longrightarrow \mathbb{R}$, $\dim(T') = n^{s+r}$?

Tensors

Ordering & Abstract Index Notation

- Both tensors are isomorphic! Both tensors share the same dimension.
- After all, tensors are made of covariant and contravariant vectors
  $T \equiv \left[T^{a_1\cdots a_r}{}_{b_1\cdots b_s}\right] e_{a_1} \otimes \cdots \otimes e_{a_r} \otimes f^{b_1} \otimes \cdots \otimes f^{b_s}$
- Linear properties of tensors guarantee that regardless of the vector type we load rows, columns, or layers with, the action of the tensors on the covariant/contravariant vectors
  $T(\xi_1, \cdots, \xi_r, X^1, \cdots, X^s) = T(\xi_{a_1} f^{a_1}, \cdots, \xi_{a_r} f^{a_r}, X^{b_1} e_{b_1}, \cdots, X^{b_s} e_{b_s})$
  gives same result.
- So all you need is consistency in loading information.
- So $A^{a_1\cdots a_r}{}_{b_1\cdots b_s} \equiv A_{b_1\cdots b_s}{}^{a_1\cdots a_r}$. Basis independend.
- Thus the 4-potential contravariant vector $A^{\ \mu} \equiv A^{\mu}$.



Construction, Transformations & Contraction

- $T \equiv T^{m_1 \cdots m_r}{}_{n_1 \cdots n_s} e_{m_1} \otimes \cdots \otimes e_{m_r} \otimes f^{n_1} \otimes \cdots \otimes f^{n_s}$.
- For $A^{m_1 \cdots m_r}{}_{n_1 \cdots n_s}$ and $B^{a_1 \cdots a_p}{}_{b_1 \cdots b_q}$ one can exploit both tensors to construct a higher $(r+p, s+q)$-ranked tensor as $\boxed{(A \otimes B)^{m_1 \cdots m_r a_1 \cdots a_p}{}_{n_1 \cdots n_s b_1 \cdots b_q} = A^{m_1 \cdots m_r}{}_{n_1 \cdots n_s} B^{a_1 \cdots a_p}{}_{b_1 \cdots b_q}}$.
- $e'_m = \Omega^a{}_m e_a \quad , \quad f'^u = \mho^m{}_a f^a \xrightarrow{f'^m e'_n = \delta^m{}_n} \left(\Omega^a{}_m\right)^{-1} \equiv \left(\Omega^{-1}\right)^m{}_a = \mho^m{}_a$.

  So, one can prove that $\boxed{\left(\Omega^a{}_m\right) \mho^m_b = \delta^a{}_b}$ [Remember the properties of mathematical groups].
- For $T^m_n = T(f'^m, e'_n) = T(\mho^m{}_a f^a, \Omega^b{}_n e_b) = \mho^m{}_a \Omega^b{}_n T(f^a, e_b) = \mho^m{}_a \Omega^b{}_n T^a_b$ [wlog].
- If a covariant and a contravariant vectors share the same index inside a tensor, they generically cancel each other, and the tensor rank becomes $(r-1, s-1)$, i.e., $\boxed{\mathcal{C}\left(T^{m_1 \cdots i \cdots m_r}{}_{n_1 \cdots i \cdots n_s}\right) = T'^{m_1 \cdots m_{r-1}}{}_{n_1 \cdots n_{s-1}}}$.

Tensors

Symmetrization

The following are basis independent, covar./contrav. independent, and row/column/layer independent.

- Symmetric tensor $S_{ab} = S_{ba} \Rightarrow \boxed{S_{(ab)} = \frac{1}{2}(S_{ab} + S_{ba})}$, Same for contravariant.

  e.g., the stress energy-momentum tensor $T_{\mu\nu} = T_{\nu\mu}$, the metric tensor $\eta_{\mu\nu} = \eta_{\nu\mu}$.

- Antisymmetric tensor $A_{ab} = -A_{ba} \Rightarrow \boxed{A_{[ab]} = \frac{1}{2}(S_{ab} - S_{ba})}$, Same for contravariant.

  e.g., the EM field strength tensor $F_{\mu\nu} = -F_{\nu\mu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

- $M_{(a|ij|b)} = \frac{1}{2}(M_{aijb} + M_{bija})$, Same for contravariant, and mixed antisymmetric have –ve.

- $M_{(abc)i} = \frac{1}{3!}(M_{abci} + M_{cabi} + M_{bcai} + M_{baci} + M_{acbi} + M_{cbai})$

- $M_i{}^{[abc]} = \frac{1}{3!}(M_i{}^{abc} + M_i{}^{cab} + M_i{}^{bca} - M_i{}^{bac} - M_i{}^{acb} - M_i{}^{cba})$

Definition & Properties

- Metric tensor is generically the covariant basis-independent bilinear map $g(0,2) : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ that measures "distances" on the manifold $T_p\mathcal{M}$ and "lengths" of contravariant vectors with the following properties:

1. $g(X,Y) = g(Y,X)$, $\forall X,Y \in T_p\mathcal{M}$, i.e., it is symmetric: $g_{ab} = g_{ba}$.

2. $g(X,Y) = 0$, $\forall Y \in T_p\mathcal{M} \iff X = 0 \in T_p\mathcal{M}$, i.e., it is "non-degenerate".

Metric Tensor

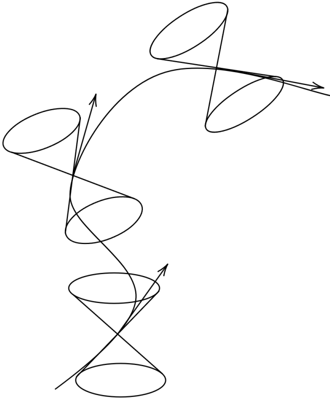
Exchanging Covarients with Contravariants and vice versa

- For $g = g_{ab} dx^a \otimes dx^b$ the basis $\{dx^a\}$ can be scaled to "orthonormal" basis $\{e_\perp^m\}$ such that g_{ab} becomes just diagonal numbers. That does NOT mean the manifold is now flat, it means we twist/deform the metric tensor (which is like a ruler) such that it matches the curved lines we study.
- $g^{\text{Riemannian}} = \text{sign}(+,+,\cdots)$, $g^{\text{pseudoRiemannian}} = \text{sign}(-,-,\cdots,+,+,\cdots)$, $g^{\text{Lorentzian}} = \text{sign}(-,+,+,+)$, where sign is the "signature" of $\mathcal{M}$.
- Define $g^{ab} \equiv g(0,2) : T_p^*\mathcal{M} \times T_p^*\mathcal{M} \to \mathbb{R}$, of course $g_{ab} \cong g^{ab}$.
- $g(X^a e_a, e_b) = X^a g(e_a, e_b) = X^a g_{ab} = X'_b$. Similarly $X'_b g^{ab} = X''^a$. But the above isomorphism demands $X^a = X''^a \Rightarrow$ $\boxed{g_{ab} X^a = X_b}$, $\boxed{g^{ab} X_a = X^b}$. Metric is unique in raising/lowering indices! [wlog]
- Non-degeneracy guarantees invertability. Therefore, one can prove

$$\boxed{g_{ab}\ g^{bc} = \delta_a{}^c} \qquad \text{and} \qquad \boxed{g_{ab}\ g^{ba} = \dim(\mathcal{M})}.$$

Curve Length

Different Types of Contravariant Vectors



- In a Lorentzian spacetime with $\text{sign}(g) = (-,+,+,+)$,
  $ds^2 \equiv -g_{ab} dx^a \otimes dx^b$
- $ds = \sqrt{-g_{ab} dx^a \otimes dx^b} = \sqrt{-dt^2 g_{ab} \frac{dx^a}{dt} \otimes \frac{dx^b}{dt}}$
- $\ell = \int dt \sqrt{-g_{ab} U^a U^b}$ [How does it look in a comoving frame?]
- Because of the signature of g_{ab}, we have different types of vectors
  i. For $g_{ab} U^a U^b < 0$, the U^a is a timelike vector.
  ii. For $g_{ab} U^a U^b = 0$, the U^a is a lightlike vector.
  iii. For $g_{ab} U^a U^b > 0$, the U^a is a spacelike vector.
- For a timelike $X \neq 0$, $g(X,Y) = 0 \Rightarrow Y$ is a spacelike.
- For a lightlike $X \neq 0$, $g(X,Y) = 0 \Rightarrow Y$ is a spacelike or lightlike.
- For a spacelike $X \neq 0$, $g(X,Y) = 0 \Rightarrow Y$ is a spacelike, lightlike, or timelike.
- Does that contradict with the non-degeneracy property of the metric? NO! WHY?

Thank You!