General Relativity Seminars

Week 2: Equivalence Principle(s) & curved spacetime

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Outline

4-acceleration

Proper time again

More on $\mathcal{A}^\mu$ can be found in M. Tsamparlis, “Special Relativity: An Introduction with 200 Problems and Solutions”, Second Edition, Springer Nature Switzerland AG, 2019, ISBN: 9783030273477, p. 191-268.

4-Acceleration & The Comoving Frames

$$\text{Back to } \eta_{\mu\nu} = \text{diag}(-1,+1,+1,+1)$$

$$\mathcal{A}^\mu := \frac{dU^\mu}{d\tau} = \gamma_u \frac{du^\mu}{d\tau} + u^\mu \frac{d\gamma_u}{d\tau} = \left[c\frac{d\gamma_u}{d\tau} \qquad (\gamma_v \frac{du^i}{d\tau} + u^i \frac{d\gamma_u}{d\tau})\right]^T,$$

$$\text{Notice } \frac{du^i}{d\tau} \neq a^i := \frac{du^i}{dt}, \text{ i.e., } \frac{d}{d\tau} \to \gamma_u \frac{d}{dt}.$$

$$\boxed{\therefore \mathcal{A}^\mu = \left[\frac{\gamma_u^4}{c} u_j a^j \qquad (\gamma_u^2 a^i + \frac{\gamma_u^4}{c^2}(u_j a^j) u^i)\right]^T}. \text{ However,}$$

$$\Lambda^\mu{}_\alpha \mathcal{A}^\alpha = \left[0 \qquad \gamma_u^2 (a^i + (\gamma_u + 1)\frac{a_j u^j}{u^2} u^i)\right]^T = \mathcal{A}^\mu_{\text{co.fr.}} \neq \mathcal{A}^\mu$$

Spacelike 4-vectore

4-Acceleration & The Comoving Frames

Additionally; one can prove $U_\mu \mathcal{A}^\mu = 0$. (Don't expand it, rely on the definition.)

$$\text{Also, } \mathcal{A}_\mu \mathcal{A}^\mu = \gamma_u^4 \left[(a_j)^2 + \frac{\gamma_u^2}{c^2} (u_j a^j)^2 \right] = \gamma_u^6 \left[\frac{(a_j)^2}{\gamma_u^2} + \frac{(a_j)^2 (u_k)^2 - (a_j u_k \epsilon^{ijk})^2}{c^2} \right]$$

$$\text{Then, } \mathcal{A}_\mu \mathcal{A}^\mu = \gamma_u^6 (a_j)^2 \left[\frac{1}{\gamma_u^2} + \frac{(u_k)^2}{c^2} - \frac{(u_k)^2}{c^2} \sin^2\theta \right]$$

$$\boxed{\therefore \mathcal{A}_\mu \mathcal{A}^\mu = \gamma_u^6 (a_j)^2 \left[1 - \frac{(u_k)^2}{c^2} \sin^2\theta \right] \geq 0} \quad , \quad \theta \text{ is between } u_j \text{ and } a_k$$

It turns out $(\mathcal{A}_\mu)_{\text{co.fr.}} (\mathcal{A}^\mu)_{\text{co.fr.}} = \mathcal{A}_\mu \mathcal{A}^\mu$ reaffirming that $\mathcal{A}^\mu$ is a spacelike vector.

Moreover, $\mathcal{A}^\mu_{\text{co.fr.}} = \mathcal{A}^\mu$ either if $u^i = 0$ (contradictory), or if $u || a$.

4-acceleration

Parallel Acceleration

Approximate Galilean Kinematics

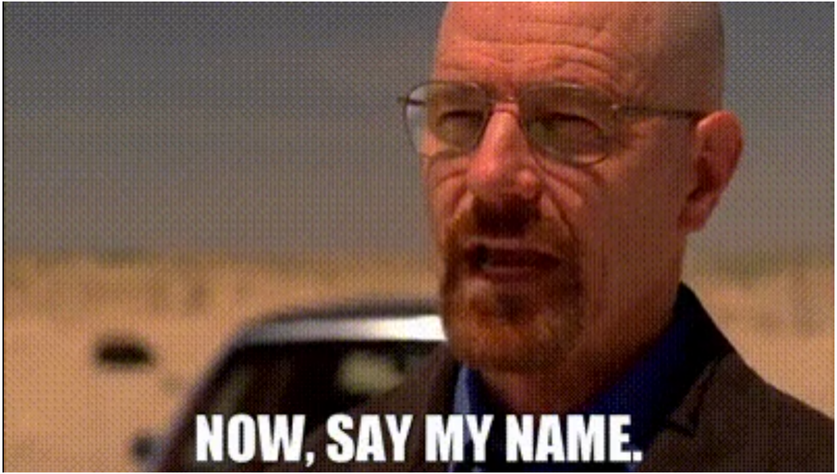
$$\text{If } u_i || a_i \Rightarrow \boxed{\mathcal{A}^i_{||} = \gamma_u^3 a^i_{||}} \Rightarrow \boxed{\mathcal{A}^\mu_{||} = \begin{bmatrix} 0 & \gamma_u^3 a^i_{||} \end{bmatrix}^T}$$

$$\frac{du}{dt} = \mathcal{A}_{||}(1 - u^2/c^2)^{3/2} \Rightarrow \boxed{u(t) = \frac{\mathcal{A}_{||} t}{\sqrt{1 + (\frac{\mathcal{A}_{||} t}{c})^2}}} \xrightarrow{\text{Galilean approx.}} u(t) \approx \mathcal{A}_{||} t, \ \ u(t_0) = 0$$

$$\text{And } \boxed{x(t) = \frac{c^2}{\mathcal{A}_{||}} \left[\sqrt{1 + \left(\frac{\mathcal{A}_{||} t}{c} \right)^2} - 1 \right] + x(t_0)} \xrightarrow{\text{Galilean approx.}} x(t) \approx \frac{1}{2} \mathcal{A}_{||} t^2 + x(t_0)$$

Now, Say My Name!

Arrays=Tensors?



Now, Say My Name!

Arrays=Tensors?

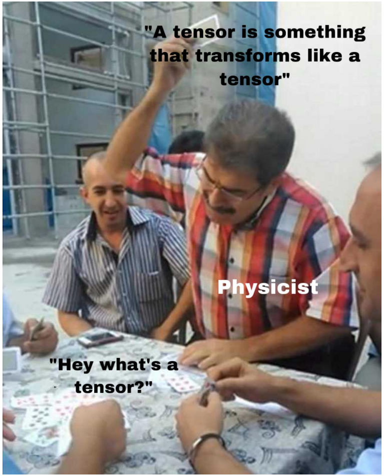
A *manifold* is a topological space that looks (i.e. it is homeomorphic to) locally (i.e. in a *patch*) like a piece of $\mathbb{R}^d$. d is the dimension of the manifold and the correspondence between the patch and the piece of $\mathbb{R}^n$ can be used to label the points in the patch by Cartesian $\mathbb{R}^n$ coordinates x^μ. In the overlap between different patches the different coordinates are consistently related by a *general coordinate transformation* (GCT) $x'^\mu(x)$. Only objects with good transformation properties under GCTs can be defined globally on the manifold. These objects are *tensors*.

T. Ortín, Gravity and Strings, CUP, 2nd Ed. (2015), p. 3

!!!!!!!!!!!

Now, Say My Name!

Arrays=Tensors?



Now, Say My Name!

Arrays=Tensors?

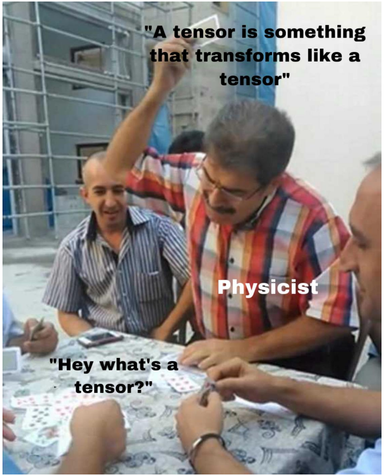
SR Lorentz Transformations on 4-Velocity

$$U'_t = \gamma_v U_t - \gamma_v \beta_v U_x$$
$$U'_x = -\gamma_v \beta_v U_t + \gamma_v U_x$$
$$U'_y = U_y \text{ and } U'_z = U_z$$

$$\begin{bmatrix} U'_t \\ U'_x \\ U'_y \\ U'_z \end{bmatrix} = \begin{bmatrix} \gamma_v & -\beta_v\gamma_v & 0 & 0 \\ -\beta_v\gamma_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_t \\ U_x \\ U_y \\ U_z \end{bmatrix}$$

Or: $$U'^{\mu} = \Lambda^{\mu}{}_{\alpha} U^{\alpha}$$

Additionally $\boxed{-U_t^2 + U_x^2 + U_y^2 + U_z^2 = -c^2}$



Now, Say My Name!

Arrays=Tensors?

$\Lambda^{\alpha}_{\ \mu}$ transfers a frame of reference $\alpha\beta\gamma\cdots$ to another $\lambda\mu\nu\cdots$. $||\Lambda|| \sim$ Jacobian.

A tensor: $U_\mu M^{\mu\nu} U_\nu = (\Lambda^{\gamma}_{\ \mu} U_\gamma)(\Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta} M^{\alpha\beta})(\Lambda^{\delta}_{\ \nu} U_\delta) = (\Lambda^{\gamma}_{\ \mu}\Lambda^{\delta}_{\ \nu}\Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta})[U_\gamma M^{\alpha\beta} U_\delta]$

Now, Say My Name!

Arrays are NOT Tensors but Tensors are Arrays

$$\mathcal{A}^{\mu} = \frac{dU^{\mu}}{d\tau}$$

This is not a tensor

in Lorentz Frames

René Magritte, "Ceci n'est pas une pipe", 1929.

$\Lambda^{\alpha}{}_{\mu}$ transfers a frame of reference $\alpha\beta\gamma\cdots$ to another $\lambda\mu\nu\cdots$. $||\Lambda|| \sim$ Jacobian.

A tensor: $U_{\mu}M^{\mu\nu}U_{\nu} = (\Lambda^{\gamma}{}_{\mu}U_{\gamma})(\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}M^{\alpha\beta})(\Lambda^{\delta}{}_{\nu}U_{\delta}) = (\Lambda^{\gamma}{}_{\mu}\Lambda^{\delta}{}_{\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta})[U_{\gamma}M^{\alpha\beta}U_{\delta}]$

Not a tensor: $\mathcal{A}^{\mu} = \frac{dU^{\mu}}{d\tau} = \partial_{\nu}U^{\mu}\frac{dx^{\nu}}{d\tau} = \Lambda^{\alpha}{}_{\nu}\partial_{\alpha}(\Lambda^{\mu}{}_{\beta}U^{\beta})(\Lambda^{\nu}{}_{\delta}\frac{dx^{\delta}}{d\tau}) \neq (\Lambda^{\alpha}{}_{\nu}\Lambda^{\mu}{}_{\beta}\Lambda^{\nu}{}_{\delta})[\frac{dx^{\delta}}{d\tau}\partial_{\alpha}U^{\beta}]$

Hyperbolic Motion

Rindler Coordinates

Very helpful to develop QFT in accelerated spacetimes considering the center of mass frame.

$$\frac{dt}{d\tau} = \sqrt{1-\left(\frac{u}{c}\right)^2} = \sqrt{1-\frac{1}{c^2}\left(\frac{\mathcal{A}_{||}t}{\sqrt{1+(\frac{\mathcal{A}_{||}t}{c})^2}}\right)^2} = \frac{1}{\sqrt{1+\left(\frac{\mathcal{A}_{||}t}{c}\right)^2}} \xRightarrow{\text{Integrate}}$$

$$\boxed{ct = \frac{c^2}{\mathcal{A}_{||}}\sinh\left(\frac{\mathcal{A}_{||}\tau}{c}\right)}$$

$$x(t) = \frac{c^2}{\mathcal{A}_{||}}\left[\sqrt{1+\left(\mathcal{A}_{||}t/c\right)^2}-1\right]+x(t_0) \xRightarrow{x(t_0)=\frac{c^2}{\mathcal{A}_{||}}} x(t) = \frac{c^2}{\mathcal{A}_{||}}\left[\sqrt{1+\left(\mathcal{A}_{||}t/c\right)^2}\right] \xRightarrow{\text{replace } t}$$

$$\boxed{x = \frac{c^2}{\mathcal{A}_{||}}\cosh\left(\frac{\mathcal{A}_{||}\tau}{c}\right)}$$

Hyperbolic Motion

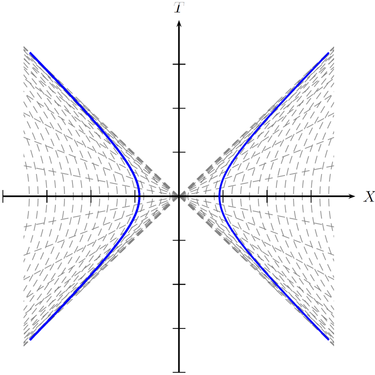
Rindler Coordinates

Set $\eta \equiv \dfrac{\mathcal{A}_{||}\tau}{c}$ as the "rapidity", and $x(t_0) \equiv \dfrac{c^2}{\mathcal{A}_{||}} := \alpha$

$$\boxed{\therefore x(t) = \alpha \cosh(\eta) \equiv X} \quad , \quad \boxed{\therefore ct = \alpha \sinh(\eta) \equiv T}$$

$$\therefore \boxed{\beta_u = \tanh(\eta)}$$

$$\boxed{-T^2 + X^2 = \alpha^2} \xrightarrow{\eta_{\mu\nu}=\text{diag}(-1,1,1,1)} \boxed{X^\mu \equiv [-T \quad X]^T}$$



Hyperbolic Motion

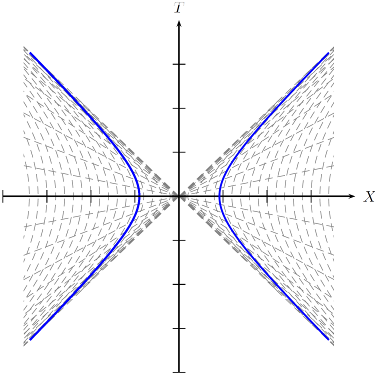
Say “au revoir” to Special Relativity!

$x(t_0)$ and τ are variables. Let's call them $\tilde{x}$ and $\tilde{t}$.

$$ct = \tilde{x}\sinh\frac{\mathcal{A}_{||}\tilde{t}}{c} \Rightarrow \boxed{cdt = d\tilde{x}\sinh\frac{\mathcal{A}_{||}\tilde{t}}{c} + (cd\tilde{t})\frac{\mathcal{A}_{||}\tilde{t}}{c^2}\cosh\frac{\mathcal{A}_{||}\tilde{t}}{c}}$$

$$x = \tilde{x}\cosh\frac{\mathcal{A}_{||}\tilde{t}}{c} \Rightarrow \boxed{dx = d\tilde{x}\cosh\frac{\mathcal{A}_{||}\tilde{t}}{c} + (cd\tilde{t})\frac{\mathcal{A}_{||}\tilde{t}}{c^2}\sinh\frac{\mathcal{A}_{||}\tilde{t}}{c}}$$

$$\begin{bmatrix} cdt \\ dx \end{bmatrix} = \begin{bmatrix} \sinh(\eta) & \cosh(\eta) \\ \cosh(\eta) & \sinh(\eta) \end{bmatrix} \begin{bmatrix} cd\tilde{t} \\ d\tilde{x} \end{bmatrix}$$

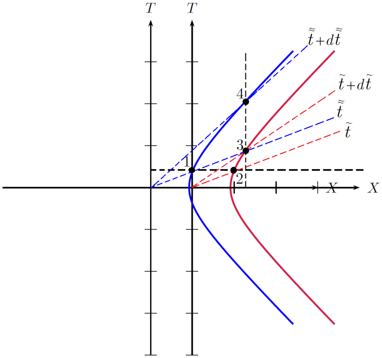


$$\therefore ds^2 = -c^2dt^2 + dx^2 \xRightarrow{\text{use } cosh^2(\eta)-sinh^2(\eta)=1} \boxed{ds^2 = -\left[\frac{\mathcal{A}_{||}\tilde{t}}{c^2}\right]^2 c^2d\tilde{t}^2 + d\tilde{x}^2}$$

If generalized this means $\boxed{\eta_{\mu\nu}(\tilde{t},\tilde{r}) := \text{diag}\left(-\left[\mathcal{A}_{||}\tilde{t}/c^2\right]^2, +1, +1, +1\right)}$

Advancing ∼ and Lagging ≈ Observers

- Red and blue observers experiencing same $\mathcal{A}_{||}$.
- Red observer gets ahead of the blue one by ℓ.
- At the same T, the blue is at 1 with $\tilde{\tilde{t}}$, meanwhile the red is at 2 with $\tilde{t}$.
- At 3, the red clock is $\tilde{t} + d\tilde{t}$ w.r.t to $FoR_{\tilde{t}}$, that corresponds to $T + dT$.
- But in the $FoR_{\tilde{\tilde{t}}}$, the red clock is at $\tilde{\tilde{t}}$, where the blue clock is still at T.
- Therefore, even if both experience the same $\mathcal{A}_{||}$, the red clock is ticking faster than the blue one!

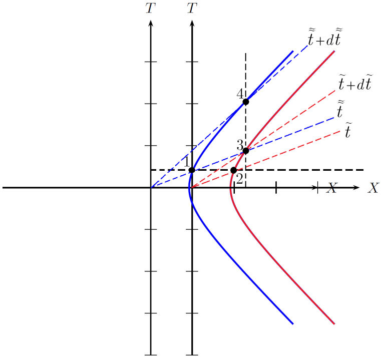


See F. J. Flores, “Communicating with Accelerated Observers in Minkowski Spacetime”, (2008) Eur. J. Phys. 29 73.

Two accelerated Observers

Advancing $\sim$ and Lagging $\approx$ Observers

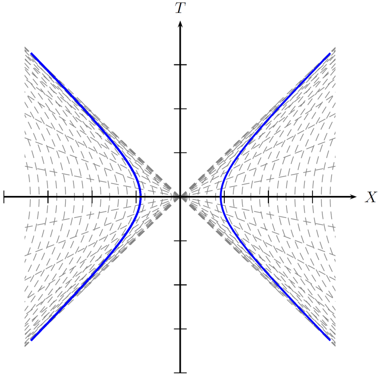
- $x_A = \alpha \cosh(\mathcal{A}_{||}\tilde{t}/c) + \ell \quad , \quad ct_A = \alpha \sinh(\mathcal{A}_{||}\tilde{t}/c).$
- $x_L = \alpha \cosh(\mathcal{A}_{||}\tilde{\tilde{t}}/c) \quad , \quad ct_L = \alpha \sinh(\mathcal{A}_{||}\tilde{\tilde{t}}/c)$
  $= x_L \tanh(\mathcal{A}_{||}\tilde{\tilde{t}}/c).$
- At 3, the A passes by $\tilde{t} + d\tilde{t}$ in $FoR_{\tilde{t}}$ and by $\tilde{\tilde{t}}$ in $FoR_{\tilde{\tilde{t}}}$.
- At 4, the L passes by $\tilde{\tilde{t}} + d\tilde{\tilde{t}}$ in $FoR_{\tilde{\tilde{t}}}$ where $x_L|_{\text{at4}} = x_A|_{\text{at3}}$.
- Thus, $c(\tilde{t} + d\tilde{t})_A = c(\tilde{\tilde{t}} + d\tilde{\tilde{t}})_L \equiv ct_L|_{x_L = x_A}$.
- Generally this happens spatial point by spatial point.
- $\boxed{\therefore ct_A = x_A \tanh(\mathcal{A}_{||}\tilde{\tilde{t}}/c)}$.



Two accelerated Observers

Advancing $\sim$ and Lagging $\approx$ Observers

- $\alpha \sinh(A_{||}\tilde{t}/c) = \left[\alpha \cosh(\mathcal{A}_{||}\tilde{t}/c) + \ell\right] \tanh(\mathcal{A}_{||}\tilde{\tilde{t}}/c).$
- $\dfrac{\alpha \sinh(A_{||}\tilde{t}/c)}{\alpha \cosh(\mathcal{A}_{||}\tilde{t}/c) + \ell} = \tanh(\mathcal{A}_{||}\tilde{\tilde{t}}/c).$
- Approx. to get $\dfrac{\mathcal{A}_{||}\tilde{t}/c}{1 + \ell/\alpha} = \mathcal{A}_{||}\tilde{\tilde{t}}/c.$
- Or: $\tilde{t}\left[1 - \dfrac{\ell}{\alpha}\right] = \tilde{\tilde{t}} \Rightarrow \boxed{\therefore \tilde{t}\left[1 - \dfrac{\mathcal{A}_{||}\ell}{c^2}\right] = \tilde{\tilde{t}}}$
- Since $\tilde{\tilde{t}} < \tilde{t} \Rightarrow \sqrt{\mathcal{A}_{||}\ell} < c$. This defines the "Rindler Horizon".



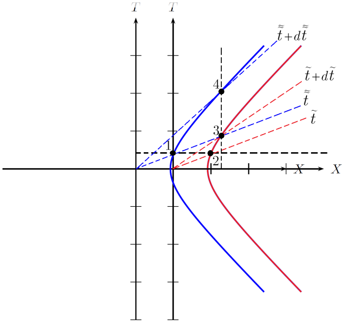
Two accelerated Observers

Light, Advancing and Lagging Observers

- In 1st week, p. 23: $p_\mu p^\mu c^4 = (m_0 c^2)^2 = E^2 - \vec{p}^2 c^2$.

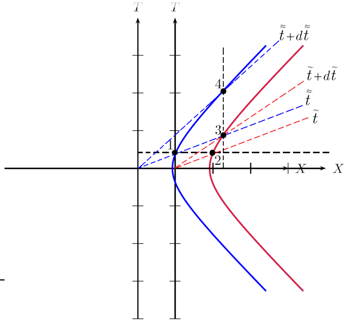
  Remember p^μ is called 4-momentum but its length equals the rest mass m_0.

- Since $m_0 = 0 \Rightarrow E = \vec{p}c \Rightarrow \vec{p} = h\nu/c$, $m \equiv h\nu/c^2$.
- For light $u_\mu = [-c \quad c]$, $p^\mu = \dfrac{1}{c^2}[h\nu \quad -h\nu]^T$.
- Thus $E_{\text{light}} = -\frac{1}{2} c u_\mu p^\mu$. $\quad [\eta_{\mu\nu} = \text{diag}(-,+,+,+)]$.
- $$u_\mu^A = \frac{d}{d\tilde{t}}\left[\alpha \sinh\left(\frac{\mathcal{A}_{||}\tau}{c}\right) \quad \alpha \cosh\left(\frac{\mathcal{A}_{||}\tau}{c}\right)\right]\Bigg|_{\tau=\tilde{t}}$$
  $$= c\left[\cosh\left(\frac{\mathcal{A}_{||}\tilde{t}}{c}\right) \quad \sinh\left(\frac{\mathcal{A}_{||}\tilde{t}}{c}\right)\right]$$
- $$\boxed{\therefore E_A = h\nu\left[\cosh\left(\frac{\mathcal{A}_{||}\tilde{t}}{c}\right) - \sinh\left(\frac{\mathcal{A}_{||}\tilde{t}}{c}\right)\right]}$$



Two accelerated Observers

Light, Advancing and Lagging Observers



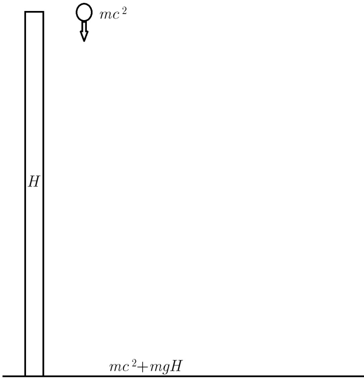
- $\therefore E_A = h\nu \cosh\left(\frac{\mathcal{A}_{||}\tilde{t}}{c}\right)\left[1 - \tanh\left(\frac{\mathcal{A}_{||}\tilde{t}}{c}\right)\right]$
- Since $\cosh(\theta) = \frac{1}{\sqrt{\text{sech}^2(\theta)}} = \frac{1}{\sqrt{1-\tanh^2(\theta)}}$
- $E_A = h\nu\sqrt{\frac{1-\tanh\left(\frac{\mathcal{A}_{||}\tilde{t}}{c}\right)}{1+\tanh\left(\frac{\mathcal{A}_{||}\tilde{t}}{c}\right)}}$, $E_L = h\nu\sqrt{\frac{1-\tanh\left(\frac{\mathcal{A}_{||}\tilde{\tilde{t}}}{c}\right)}{1+\tanh\left(\frac{\mathcal{A}_{||}\tilde{\tilde{t}}}{c}\right)}}$
- Use $\tanh(\theta) \approx \theta$ and $\tilde{\tilde{t}} = (1 - \mathcal{A}_{||}\ell/c^2)\tilde{t}$ s.t. $\boxed{E_L - E_A = h\nu\frac{\mathcal{A}_{||}\ell}{c^2}}$

  i.e., when clock ticks faster, energy becomes lower and vice versa!

Gravity Effect on Light

Pound-Rebka Experiment & Gravitational Redshift

- At the top $E = mc^2$.
- At the bottom $E = mc^2 + mgH$.
- At bottom $E(m) \to E = h\nu$ then goes up.
- If gravity had NOT affected $E(\nu) \to E(\nu')$ s.t. $E(\nu') = mc^2$, then we would have invented a machine producing infinite energy!!
- $\dfrac{E_{\text{top}}}{E_{\text{bottom}}} = \dfrac{h\nu'}{h\nu} = \dfrac{mc^2}{mc^2 + mgH} \Rightarrow \boxed{\therefore \dfrac{\nu'}{\nu} = 1 - \dfrac{gH}{c^2}}$
- Generally in "uniform" $\mathcal{A}_{||}$, $\boxed{\dfrac{\nu'}{\nu} = 1 - \dfrac{\Delta\phi^G}{c^2}}$ this is the "gravitational redshift".



Principle(s) of Equivalence

- Weak Equivalence Principle: The uniform acceleration is indistinguishable from a uniform gravitational field, i.e., $F = m_I \mathcal{A}_{||} = m_G g \xrightarrow{g=\mathcal{A}_{||}} m_I = m_G$. Therefore, freely falling test objects follow the same trajectories if they share the same initial conditions.
- Einstein equivalence principle: Beside the weak one, in a local inertial frame, the results of all non gravitational experiments will be indistinguishable from the results of the same experiments performed in an inertial frame in Minkowski spacetime.
- Strong Equivalence Principle : Einstein principle is valid also for massive objects, i.e., exerting gravity force on its components, and thus Newton's constant G must be constant!
- In EM, $\Box A^\mu = \mu_0 J^\mu + \cdots$. Can gravity be in form of $\Box \phi = 4\pi G \rho$? NO!

Principle(s) of Equivalence

- For more on why ρ is not enough to describe gravity, see M. Janssen, J. D. Norton, J. Renn, T. Sauer, J. Stachel, Eds., "The Genesis of General Relativity: Sources and Interpretations", Volume 1, Springer Dordrecht, 2007, ISBN:9781402039997, p.53-111, where it discusses how Max von Laue introduced stress-energy-momentum tensor to gravity between 1911-1912 before the advent of GR itself. Or you may spend your summer reading the 4-volume set!
- For more on why gravity can neither be described by a scalar ϕ^G nor by a 4-vector A^μ, see R. Feynman, F. Morinigo, W. Wagner, and B. Hatfield, Eds., "Feynman Lectures on Gravitation", Westview Press, 2002, p.29-31.

Thank You!