General Relativity Seminars

Week 1: Prelude to the Special Theory of Relativity & Lorentzian spacetime

Hassan Alshal

Outline

1. Michelson-Morley Experiment
2. Lorentz Transformations
3. “Array” Structure of Electrodynamics
4. Noether Symmetries & Equations of Motion

Michelson-Morley Experiment

Constancy of the Speed of Light

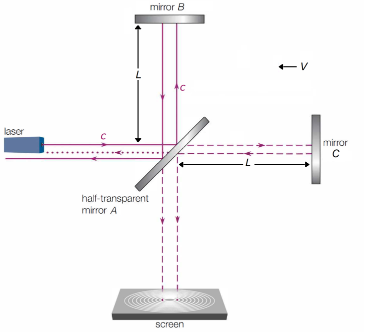
An observer on Earth describing speed of light c' through both arms, thinking it's affected by "Aether" winds with velocity $-v\hat{e}_x$, where speed of light in Aether frame is c.

- From A to C: $c' = c - v \quad , \quad t_{AC} = L/(c-v)$.
- From C to A: $c' = c + v \quad , \quad t_{CA} = L/(c+v)$.

$$\therefore t_{ACA} = \frac{2L}{c(1 - v^2/c^2)}$$

- From A to B: $c' = \sqrt{c^2 - v^2} \quad , \quad t_{AB} = L/\sqrt{c^2 - v^2}$.
- From B to A: $c' = \sqrt{c^2 - v^2} \quad , \quad t_{BA} = L/\sqrt{c^2 - v^2}$.

$$\therefore t_{ABA} = \frac{2L}{c\sqrt{1 - v^2/c^2}}$$



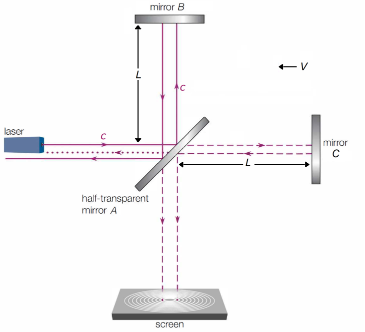
Michelson-Morley Experiment

Constancy of the Speed of Light

$$\Delta t_{arms} = t_{ACA} - t_{ABA} = \frac{2L}{c}\left[\frac{1}{1-v^2/c^2} - \frac{1}{\sqrt{1-v^2/c^2}}\right]$$

$$\approx \frac{2L}{c}\left[1 + v^2/c^2 - 1 - \frac{1}{2}v^2/c^2\right] = (v^2/c^2)\frac{L}{c} = \beta_v^2\frac{L}{c}$$

If the experiment is rotated 90°, then it's "expected" that $\Delta t_{rotated} = -\beta_v^2\frac{L}{c}$, and thus $\Delta t_{total} = 2\beta_v^2\frac{L}{c}$, or $\Delta\lambda = 2\beta_v^2 L$ corresponds to $n = 2\beta_v^2\frac{L}{\lambda}$. But fringe shift had never been found even after repeating the experiment 6 months later!!!



© 2010 Encyclopædia Britannica, Inc.

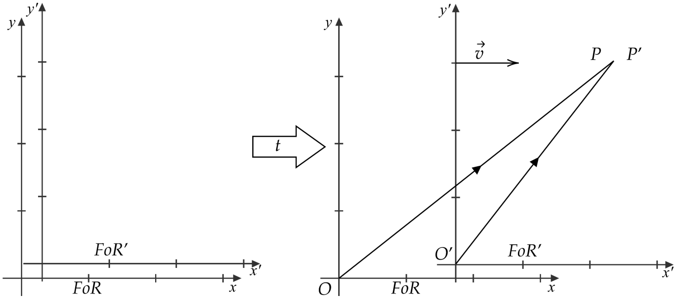
Constancy of the Speed of Light

Explanations:

- Earth is at rest with aether. (Rejected as is against Leibniz relationalism. See "Absolute and Relational Space and Motion: Classical Theories" article on Stanford Encyclopedia of Philosophy).
- Earth drags some aether atoms around it s.t. they capture the speed of earth. (Rejected by the 1893 Lodge experiment).
- FitzGerald 1889 ad hoc: Objects inside aether contract by $\sqrt{1-v^2/c^2}$ factor to compensate for t_{ACA} s.t $\Delta t_{arms}=0$. (Despite being true, FitzGerald's reasoning was incorrect as rulers should also contract; it's unfalsifiable).
- In addition to the Principle of Relativity, the velocity of light is the same with respect to all inertial frames, and there is no such thing called aether! (Einstein-Poincaré assumption).

Lorentz Transformations

Space & Time Intertwined



- At $t = 0$, FoR and FoR' are identical, and a light source emits a wave.
- After a time t, O detects the wave at P while O' detect the same wave at P' .
- $\vec{OP}^2 = (ct)^2 = r^2 \quad , \quad \vec{O'P'}^2 = (ct')^2 = r'^2 \;\Rightarrow\; 0 = -(ct)^2 + r^2 = -(ct')^2 + r'^2$.
- To simplify the calculations set $y = y' \quad , \quad z = z'$ s.t. $\vec{OP}^2 - \vec{O'P'}^2 = r^2 - r'^2$.

Lorentz Transformations

Space & Time Intertwined

- O suggests $x' = ax + bt$. When $x' = 0 \Rightarrow x/t = -b/a = v \Rightarrow \boxed{x' = a(x - vt)}$.
- O' suggests $x = fx' + gt'$. When $x = 0 \Rightarrow -x'/t' = g/f = v \Rightarrow \boxed{x = f(x' + vt')}$.
- Combine the two results s.t. $x = f\left[a(x - vt) + vt'\right] \Rightarrow \boxed{t' = a\left[t - \frac{1}{v}(1 - \frac{1}{af})x\right]}$.
- Since $-(ct')^2 + x'^2 = 0$, one can prove that

$$x^2\left[a^2 - \frac{a^2c^2}{v^2}(1 - \frac{1}{af})^2\right] + xt\left[-2a^2v + \frac{2a^2c^2}{v}(1 - \frac{1}{af})\right] + t^2\left[a^2v^2 - a^2c^2\right] = 0$$

then compare the last result with $-(ct)^2 + x^2 = 0$ to get

$$\boxed{a = f = 1/\sqrt{1 - v^2/c^2} \equiv \gamma_v}$$

Lorentz Transformations

Lorentz-Voigt Group

$$ct' = \gamma_v(ct - \beta x)$$
$$x' = \gamma_v(x - \beta ct)$$
$$y' = y \text{ and } z' = z$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma_v & -\beta_v\gamma_v & 0 & 0 \\ -\beta_v\gamma_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

Or: $$x'^{\mu} = \Lambda^{\mu}{}_{\alpha}\, x^{\alpha}$$

A set of transformations G, containing Λ's and a special $\Lambda_0 := id$, forms a group if:

- for every Λ there exists a Λ^{-1} s.t. $\Lambda * \Lambda^{-1} = \Lambda^{-1} * \Lambda = id$.
- $(\Lambda * \Lambda') * \Lambda'' = \Lambda * (\Lambda' * \Lambda'')$

Space & Time in Special Relativity

In Galilean "affine" space $dr^2 = dx^2 + dy^2 + dz^2 = dx'^2 + dy'^2 + dz'^2$ is "invariant" under Galilean transformations $x' = x \pm vt$. But under Lorentz transformations one can prove that dr^2 is not invariant. But $\vec{OP}$ and $\vec{O'P'}$ definitions inspire studying the "Lorentz invariance" of

$$\boxed{ds^2 := -c^2 dt^2 + dr^2 = -c^2 dt'^2 + dr'^2}$$

which is considered a Lorentz scalar, a special type of Lorentz "arrays".

When two events happen at the same place in different times w.r.t a stationary FoR, i.e., $dr = 0$ and $dt \neq 0$, then the non-stationary FoR' sees time interval as

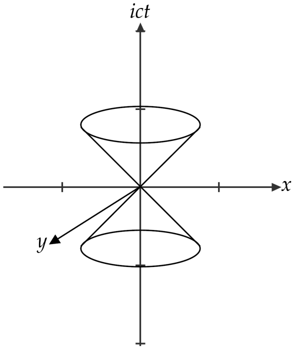
$$\boxed{dt' = \gamma_{\vec{u}'} dt} \quad , \quad \vec{u}'^2 = (\frac{dr'}{dt'})^2$$

FoR' considers its dt' "the proper time" $d\tau$. Every frame has its own proper time.

Minkowski Spacetime

$$ds^2 = (icdt)^2 + (dr)^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

- $ds^2 < 0 \Rightarrow$ timelike intervals with past and future cones, $u < c$ for massive objects.
- $ds^2 = 0 \Rightarrow$ lightlike intervals on the surfaces of the past and the future cones, $u = c$ for light-like particles.
- $ds^2 > 0 \Rightarrow$ spacelike intervals elsewhere, $u > c$ for "ghosts", "tachyons" and non-causally ordered events.



More importantly, u is what defines this Lorentzian geometry. Next lectures we see velocities $\vec{u}$, not distances $\vec{r}$, are the underpinnings of the spacetime.

Lorentz Transformations

Velocity transformations

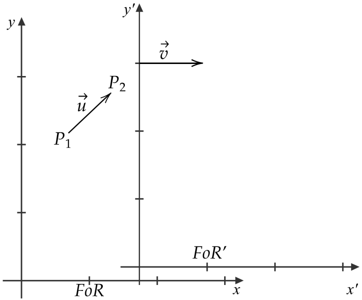
For simplicity set $y = y'$ and $\vec{v} = v\hat{e}_x$

<u>*FoR*</u>: $P_1 \to P_2$, i.e., $(t, x, y) \to (t + dt, x + dx, y + dy)$

<u>*FoR′*</u>: $P'_1 \to P'_2$, i.e., $(t', x', y') \to (t' + dt', x' + dx', y' + dy')$

$cdt' = \gamma_v(cdt - \beta dx)$

$dx' = \gamma_v(dx - \beta cdt)$



One can use the above transformations to <u>prove</u>:

$$u'_x = \frac{dx'}{dt'} = \frac{u_x - v}{1 - u_x v/c^2} \quad \text{or} \quad u_x = \frac{dx}{dt} = \frac{u'_x + v}{1 + u'_x v/c^2}$$

$$u'_y = \frac{dy'}{dt'} = \frac{u_y}{\gamma_v(1 - u_x v/c^2)} \quad \text{or} \quad u_y = \frac{dy}{dt} = \frac{u'_y}{\gamma_v(1 + u'_x v/c^2)}$$

But this does NOT look like $u'_x = \Lambda u_x$. Also, we expect $u'_y = u_y$!

Velocity transformations

- We ignored the FoR_p' of the particle itself!
- In FoR_p' of the particle $dx_p' = 0$ and $dt_p' = d\tau$, i.e., $ds^2 = -c^2 d\tau^2$

$$-c^2 d\tau^2 = -dt^2 \left[c^2 - (\frac{dx}{dt})^2 \right] = -c^2 dt^2 (1 - u_x^2/c^2) \Rightarrow \boxed{d\tau = \gamma_{u_x}^{-1} dt}$$

- $d\tau$ is a Lorentz invariant, so define $U_x \equiv U_x(\tau) = \dfrac{dx}{d\tau}$ and $U_x' \equiv U_x'(\tau) = \dfrac{dx'}{d\tau}$.
- $U_x = \dfrac{dx}{d\tau} = \dfrac{dx}{dt}\dfrac{dt}{d\tau} = \gamma_{u_x} u_x \Rightarrow \boxed{U_x = \gamma_{u_x} u_x}$.
- Similarly $\boxed{U_x' = \gamma_{u_x'} u_x'}$.

Lorentz Transformations

Velocity transformations

- $U'_x = \frac{dx'}{dt}\frac{dt}{d\tau} = \gamma_v \frac{1}{dt}(dx - \beta_v c dt)\gamma_{u_x} = -\gamma_v\beta_v(\gamma_{u_x}c) + \gamma_v U_x = -\gamma_v\beta_v U_t + \gamma_v U_x$

$$\boxed{\therefore U'_x = -\gamma_v\beta_v U_t + \gamma_v U_x}$$

- $\boxed{U'_t = \gamma_{u'_x} c}$, so how does $\gamma_{u'_x}$, and consequently U'_t, transform between frames?
- $\frac{dt}{dt'} = \frac{d}{dt'}\left[\gamma_v(t' + \frac{x'v}{c^2})\right] = \gamma_v(1 + \frac{u'_x v}{c^2}) \xRightarrow{\text{fill in the steps}} \boxed{\frac{dt}{dt'} = \frac{1}{\gamma_v(1 - u_x v/c^2)}}$
- $\gamma_{u'_x} = \frac{1}{\sqrt{1-(u'_x/c)^2}} = \frac{1}{\sqrt{1-(\frac{dx'}{dt}\frac{dt}{dt'}/c)^2}} \xRightarrow{\text{do the steps}} \boxed{\gamma_{u'_x} = \frac{1}{c}\gamma_v(c - \beta_v u_x)\gamma_{u_x}}$

$$\therefore \boxed{U'_t = \gamma_v U_t - \gamma_v\beta_v U_x}$$

- And one can check that $\boxed{U'_y = U_y}$.

“Array” Structure of Electrodynamics

Maxwell’s Equations & Lorentz Force

Interested in SR history? W-T. Ni, One Hundred Years of General Relativity: From Genesis and Empirical Foundations to Gravitational Waves, Cosmology and Quantum Gravity, Volume 1, World Scientific, ISBN: 9789814635134, p. 1-83.

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0}\quad,\quad \vec{\nabla}\cdot\vec{B}=0\quad,\quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\quad,\quad \vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\vec{F}=q(\vec{E}+\vec{v}\times\vec{B})\Rightarrow \boxed{\vec{f}=\rho\vec{E}+\vec{J}\times\vec{B}}$$

$$\vec{f}=\epsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E}+\frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B}-\epsilon_0\frac{\partial\vec{E}}{\partial t}\times\vec{B}$$

$$\text{Use:}\quad \frac{\partial(\vec{E}\times\vec{B})}{\partial t}=\frac{\partial\vec{E}}{\partial t}\times\vec{B}+\vec{E}\times\frac{\partial\vec{B}}{\partial t}=\frac{\partial\vec{E}}{\partial t}\times\vec{B}-\vec{E}\times(\vec{\nabla}\times\vec{E})$$

$$\text{s.t.:}\quad \vec{f}=\epsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E}+\frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B}-\epsilon_0\frac{\partial(\vec{E}\times\vec{B})}{\partial t}-\epsilon_0\vec{E}\times(\vec{\nabla}\times\vec{E})$$

$$\text{or:}\quad \boxed{\vec{f}=\epsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E}-\vec{E}\times(\vec{\nabla}\times\vec{E})\right]+\frac{1}{\mu_0}\left[(\vec{\nabla}\cdot\vec{B})\vec{B}-\vec{B}\times(\vec{\nabla}\times\vec{B})\right]-\epsilon_0\mu_0\frac{\partial(\vec{S})}{\partial t}}$$

"Array" Structure of Electrodynamics

Maxwell's Equations & Lorentz Force

where $\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$ is the Poynting vector.

$$\vec{f} = \epsilon_0 \left[(\vec{\nabla} \cdot \vec{E})\vec{E} - \vec{E} \times (\vec{\nabla} \times \vec{E})\right] + \frac{1}{\mu_0}\left[(\vec{\nabla} \cdot \vec{B})\vec{B} - \vec{B} \times (\vec{\nabla} \times \vec{B})\right] - \epsilon_0\mu_0\frac{\partial \vec{S}}{\partial t}$$

Use the identity:

$$\vec{V} \times (\vec{\nabla} \times \vec{V}) = \frac{1}{2}\vec{\nabla}(\vec{V} \cdot \vec{V}) - (\vec{V} \cdot \vec{\nabla})\vec{V}$$

such that:

$$\boxed{\vec{f} = \epsilon_0 \left[(\vec{\nabla} \cdot \vec{E})\vec{E} + (\vec{E} \cdot \vec{\nabla})\vec{E}\right] + \frac{1}{\mu_0}\left[(\vec{\nabla} \cdot \vec{B})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{B}\right] - \frac{1}{2}\vec{\nabla}(\epsilon_0\vec{E} \cdot \vec{E} + \frac{1}{\mu_0}\vec{B} \cdot \vec{B}) - \epsilon_0\mu_0\frac{\partial \vec{S}}{\partial t}}$$

"Array" Structure of Electrodynamics

Arrays Inside Vectors & Einstein Summation Notation

$$\boxed{(\vec{V}\cdot\vec{\nabla})\vec{V} = \vec{V}(\vec{\nabla}\otimes\vec{V})}$$

$$c = \sum_{i=1}^{n=3} a_i b^i = a_i b^i = a \bullet b \qquad , \qquad \overleftrightarrow{T} = \sum_{i=1}^{n=3}\sum_{j=1}^{n=3} a_i b^j = a_i b^j = a \otimes b$$

$$[\partial_x \quad \partial_y \quad \partial_z] = \vec{\nabla} \equiv \frac{\partial}{\partial x^i} = \partial_i \quad , \quad i = 1,2,3 \quad , \quad \frac{\partial}{\partial t} = \partial_t$$

$$\delta_i^{\ j} = \mathrm{diag}(1,1,1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that a diagonal tensor $\overleftrightarrow{A} = a_i\ \delta_k^{\ j}\ a^k = a \otimes a$

and its trace $\mathrm{Tr}(\overleftrightarrow{A}) = a_i\ \delta_k^{\ i}\ a^k = a \bullet a$

Arrays Inside Vectors & Einstein Summation Notation

Double check $\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V} = \vec{V}\left(\vec{V}\otimes\vec{\nabla}\right)$

$$\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V} = (V_i\partial^i)V_j =$$

$$\left(\begin{bmatrix} V_x & V_y & V_z \end{bmatrix}\begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix}\right)\begin{bmatrix} V_x & V_y & V_z \end{bmatrix} =$$

$$(V_x\partial_x + V_y\partial_y + V_z\partial_z)\begin{bmatrix} V_x & V_y & V_z \end{bmatrix} =$$

$$\begin{bmatrix} V_x\partial_x V_x + V_y\partial_y V_x + V_z\partial_z V_x \\ V_x\partial_x V_y + V_y\partial_y V_y + V_z\partial_z V_y \\ V_x\partial_x V_z + V_y\partial_y V_z + V_z\partial_z V_z \end{bmatrix}^T$$

$$\vec{V}\left(\vec{V}\otimes\vec{\nabla}\right) = V_i(\partial^i V_j) =$$

$$\begin{bmatrix} V_x & V_y & V_z \end{bmatrix}\left(\begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix}\begin{bmatrix} V_x & V_y & V_z \end{bmatrix}\right) =$$

$$\begin{bmatrix} V_x & V_y & V_z \end{bmatrix}\begin{bmatrix} \partial_x V_x & \partial_x V_y & \partial_x V_z \\ \partial_y V_x & \partial_y V_y & \partial_y V_z \\ \partial_z V_x & \partial_z V_y & \partial_z V_x \end{bmatrix} =$$

$$\begin{bmatrix} V_x\partial_x V_x + V_y\partial_y V_x + V_z\partial_z V_x \\ V_x\partial_x V_y + V_y\partial_y V_y + V_z\partial_z V_y \\ V_x\partial_x V_z + V_y\partial_y V_z + V_z\partial_z V_z \end{bmatrix}^T$$

“Array” Structure of Electrodynamics

Lorentz Force in Einstein Summation Notation

Therefore, $\vec{f} = \rho\vec{E} + \vec{J} \times \vec{B}$ is developed into

$$\boxed{\vec{f} = \epsilon_0 \left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E}\right] + \frac{1}{\mu_0}\left[(\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B}\right] - \frac{1}{2}\vec{\nabla}(\epsilon_0\vec{E}\cdot\vec{E} + \frac{1}{\mu_0}\vec{B}\cdot\vec{B}) - \epsilon_0\mu_0\frac{\partial\vec{S}}{\partial t}}$$

then becomes

$$f_i = \epsilon_0\left[\partial_j(E_iE^j) - \frac{1}{2}\partial_j(\delta_i{}^jE_kE^k)\right] + \frac{1}{\mu_0}\left[\partial_j(B_iB^j) - \frac{1}{2}\partial_j(\delta_i{}^jB_kB^k)\right] - \epsilon_0\mu_0\partial_tS_i$$

$$\boxed{-f_i = \epsilon_0\mu_0\partial_tS_i - \partial_j\left\{\epsilon_0\left[(E_iE^j) - \frac{1}{2}(\delta_i{}^jE_kE^k)\right] + \frac{1}{\mu_0}\left[(B_iB^j) - \frac{1}{2}(\delta_i{}^jB_kB^k)\right]\right\}}$$

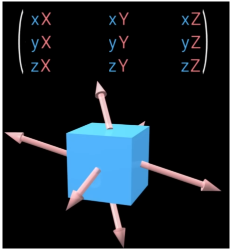
“Array” Structure of Electrodynamics

Maxwell Stress “Array” & the 4–Vector

Introduce $\boxed{\mathfrak{s}_i{}^j = \epsilon_0 (E_i E^j - \frac{1}{2}\delta_i{}^j E^2) + \frac{1}{\mu_0}(B_i B^j - \frac{1}{2}\delta_i{}^j B^2)}$

or

$$\overleftrightarrow{\mathfrak{S}} = \begin{bmatrix} \frac{1}{2}(\epsilon_0 E_x^2 + \frac{1}{\mu_0} B_x^2) & \epsilon_0 E_x E_y + \frac{1}{\mu_0} B_x B_y & \epsilon_0 E_x E_z + \frac{1}{\mu_0} B_x B_z \\ \epsilon_0 E_y E_x + \frac{1}{\mu_0} B_y B_x & \frac{1}{2}(\epsilon_0 E_y^2 + \frac{1}{\mu_0} B_y^2) & \epsilon_0 E_y E_z + \frac{1}{\mu_0} B_y B_z \\ \epsilon_0 E_z E_x + \frac{1}{\mu_0} B_z B_x & \epsilon_0 E_z E_y + \frac{1}{\mu_0} B_z B_y & \frac{1}{2}(\epsilon_0 E_z^2 + \frac{1}{\mu_0} B_z^2) \end{bmatrix}$$



And with $\vec{\nabla} = \begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix}$, we get $\boxed{-\vec{f} = \epsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t} - \vec{\nabla} \bullet \overleftrightarrow{\mathfrak{S}}}$

How to accommodate $\vec{S}$ inside $\overleftrightarrow{\mathfrak{S}}$?

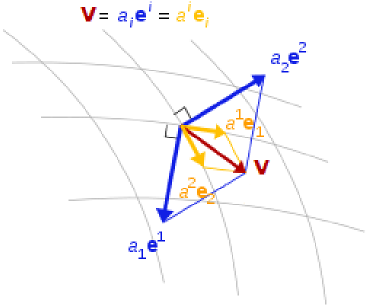
Invent a 4×4 “Array”, i.e., we expand the $3D$ space to $4D$ spacetime!

Expand $\vec{\nabla}$ to accommodate $\frac{\partial}{\partial t}$ s.t. we define the “Lorentz derivative”

$$\boxed{\partial_\mu = \begin{bmatrix} \partial_t & \partial_x & \partial_y & \partial_z \end{bmatrix} = \begin{bmatrix} \partial_0 & \partial_i \end{bmatrix}} \quad , \mu = 0, 1, 2, 3$$

“Array” Structure of Electrodynamics

4-vector & Special Relativity Metric “Array”



$V_\mu V^\mu = \overleftrightarrow{M}^{\mu\nu} V_\mu V_\nu = \overleftrightarrow{M}_{\mu\nu} V^\mu V^\nu = \overleftrightarrow{M}_\nu{}^\mu V_\mu V^\nu = ||V||^2$

e.g., in $3D$ space: $E_i E^i = E_i \delta_j{}^i E^j = E_x^2 + E_y^2 + E_z^2$

Generally $\langle V, W \rangle \equiv V_\mu \overleftrightarrow{M}^{\mu\nu} W_\nu = \vec{V} \bullet \vec{W}$

In $4D$ SR: $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + dx^2 + dy^2 + dz^2$

$$\text{Thus} \quad \overleftrightarrow{M} := \eta^{\mu\nu} = \eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} = \text{diag}(-1, 1, 1, 1)$$

In particle physics: $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \Rightarrow \eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(+1, -1, -1, -1)$

$$\text{so} \quad \boxed{\partial_\mu = \begin{bmatrix} \partial_t & \partial_x & \partial_y & \partial_z \end{bmatrix} = \begin{bmatrix} \partial_0 & \partial_i \end{bmatrix}} \quad , \quad i = 1, 2, 3$$

$$\text{or} \quad \partial^\mu = \partial_\nu \eta^{\mu\nu} = \begin{bmatrix} \partial^t & -\partial^x & -\partial^y & -\partial^z \end{bmatrix}^T = \begin{bmatrix} \partial^0 & -\partial^i \end{bmatrix}^T \quad , \quad \mu = 0, 1, 2, 3$$

Stress-Energy-Momentum "Array"

$$\text{Therefore, } -\vec{f} = \epsilon_0\mu_0 \frac{\partial \vec{S}}{\partial t} - \vec{\nabla} \bullet \overleftrightarrow{\mathfrak{S}} \text{ becomes } \boxed{-f_\mu = \partial_\nu T_\mu{}^\nu}, \text{ where } f_\mu = \begin{bmatrix} f_t & f_x & f_y & f_z \end{bmatrix},$$

$$\text{and} \qquad T_\mu{}^\nu = \left[\begin{array}{c|c} U_{em} & p_i \\ \hline p^i & \mathfrak{S}_i{}^j \end{array}\right] = \begin{bmatrix} U_{em} & p_x & p_y & p_z \\ p^x & \mathfrak{s}_x{}^x & \mathfrak{s}_y{}^x & \mathfrak{s}_z{}^x \\ p^y & \mathfrak{s}_x{}^y & \mathfrak{s}_y{}^y & \mathfrak{s}_z{}^y \\ p^z & \mathfrak{s}_x{}^z & \mathfrak{s}_y{}^z & \mathfrak{s}_z{}^z \end{bmatrix}$$

If fully expanded,

$$T_\mu{}^\nu = \begin{bmatrix} \frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2) & \mu_0\epsilon_0 S_x & \mu_0\epsilon_0 S_y & \mu_0\epsilon_0 S_z \\ \mu_0\epsilon_0 S^x & \frac{1}{2}(\epsilon_0 E_x^2 + \frac{1}{\mu_0}B_x^2) & (\epsilon_0 E_x E_y + \frac{1}{\mu_0}B_x B_y) & (\epsilon_0 E_x E_z + \frac{1}{\mu_0}B_x B_z) \\ \mu_0\epsilon_0 S^y & (\epsilon_0 E_y E_x + \frac{1}{\mu_0}B_y B_x) & \frac{1}{2}(\epsilon_0 E_y^2 + \frac{1}{\mu_0}B_y^2) & (\epsilon_0 E_y E_z + \frac{1}{\mu_0}B_y B_z) \\ \mu_0\epsilon_0 S^z & (\epsilon_0 E_z E_x + \frac{1}{\mu_0}B_z B_x) & (\epsilon_0 E_z E_y + \frac{1}{\mu_0}B_z B_y) & \frac{1}{2}(\epsilon_0 E_z^2 + \frac{1}{\mu_0}B_z^2) \end{bmatrix}$$

$$\text{e.g., } \begin{bmatrix} \partial_t & \partial_x & \partial_y & \partial_z \end{bmatrix} \cdot T_x{}^\mu = \begin{bmatrix} \partial_t p_x & +\partial_i \mathfrak{s}_x{}^i \end{bmatrix} = -f_x$$

Electromagnetic Field Strength "Array"

But $\vec{E} = -\vec{\nabla}\phi - \sqrt{\mu_0\epsilon_0}\,\partial_t\vec{A}$, $\vec{B} = \vec{\nabla}\times\vec{A}$

Then, $A_\mu = \begin{bmatrix}\phi & -A_i\end{bmatrix}$, or $A^\mu = \begin{bmatrix}\phi & A^i\end{bmatrix}^T$

Therefore $$\boxed{T_\mu{}^\nu = \frac{1}{\mu_0}\left[F_\mu{}^\alpha\;\eta_\alpha{}^\beta\;F_\beta{}^\nu + \frac{1}{4}F_\alpha{}^\beta\;\eta_\mu{}^\nu\;F_\beta{}^\alpha\right]}$$

where $\boxed{F_\mu{}^\nu = \partial_\mu A^\nu - \partial^\nu A_\mu}$ & $\boxed{\eta_\mu{}^\nu = \mathrm{diag}(1,1,1,1)}$

The fully "contravariant"/"covariant" structure

$$F^{\mu\nu} = \begin{bmatrix} 0 & -\sqrt{\mu_0\epsilon_0}E_x & -\sqrt{\mu_0\epsilon_0}E_y & -\sqrt{\mu_0\epsilon_0}E_z \\ \sqrt{\mu_0\epsilon_0}E_x & 0 & -B_z & B_y \\ \sqrt{\mu_0\epsilon_0}E_y & B_z & 0 & -B_x \\ \sqrt{\mu_0\epsilon_0}E_z & -B_y & B_x & 0 \end{bmatrix} = -F_{\mu\nu}$$

4-vectors, Lagrangian(s) as Function(s) in EM “Arrays”

$$\vec{F} := \frac{d\vec{p}}{d\tau} = q\left[-\vec{\nabla}(\phi/\sqrt{\mu_0\epsilon_0}) - \partial_t\vec{A} + \vec{u}\times(\vec{\nabla}\times\vec{A})\right]$$
$$= q\left[-\vec{\nabla}(\phi/\sqrt{\mu_0\epsilon_0}) - \partial_t\vec{A} + \vec{\nabla}(\vec{u}\cdot\vec{A}) - (\vec{u}\cdot\vec{\nabla})\vec{A}\right]$$

$$\text{But:}\quad \frac{d\vec{A}}{d\tau} = \partial_t\vec{A} + (\vec{u}\cdot\vec{\nabla})\vec{A}$$

$$\text{Then:}\quad \frac{d\vec{p}}{d\tau} = q\left[-\vec{\nabla}\left((\phi/\sqrt{\mu_0\epsilon_0}) - \vec{u}\cdot\vec{A}\right) - \frac{d\vec{A}}{d\tau}\right],$$

$$\frac{d\vec{P}_{\text{total}}}{d\tau} = \frac{d(\vec{p}+q\vec{A})}{d\tau} = q\left[-\vec{\nabla}\left((\phi/\sqrt{\mu_0\epsilon_0}) - \vec{u}\cdot\vec{A}\right)\right] \Rightarrow \frac{d(\vec{P}_{\text{total}} - q\vec{A})}{d\tau} = \frac{d\vec{p}}{d\tau} = \frac{dL}{d\vec{r}}$$

$$\text{Remember}\quad \vec{F} = -\vec{\nabla}V \;\;\&\; \int\vec{F}d\vec{r} = \Delta T$$

$$\text{Thus, we introduce:}\quad \boxed{L = T - V = \frac{\vec{p}^{\,2}}{2m} - q(\phi/\sqrt{\mu_0\epsilon_0}) + q\vec{u}\cdot\vec{A}}\;, \text{ where }\; \vec{p} = m\vec{u},$$

4-vectors and Lagrangian(s) as Function(s) in EM "Arrays"

In particle physics, γ_u is absorbed inside m. Remember $U^\mu \neq \vec{u}$ but $U^i = \gamma_{u^i} u^i$.

$V(\vec{v}, \vec{A})$ can be generalized as function of 4-vectors: $V = u_\mu A^\mu$, where:

$$u_\mu = \begin{bmatrix} \frac{1}{\sqrt{\mu_0\epsilon_0}} & -\vec{u} \end{bmatrix}, \text{ or: } u^\mu = \begin{bmatrix} \frac{1}{\sqrt{\mu_0\epsilon_0}} & \vec{u} \end{bmatrix}^T$$

$$\times\sqrt{\mu_0\epsilon_0}\, m \text{ to get: } p_\mu = \begin{bmatrix} m & -\sqrt{\mu_0\epsilon_0}\,\vec{p} \end{bmatrix}, \text{ or: } p^\mu = \begin{bmatrix} m & \sqrt{\mu_0\epsilon_0}\,\vec{p} \end{bmatrix}^T$$

$$\text{s.t. } \frac{dp^\mu}{d\tau} = q u_\nu F^{\mu\nu} = -\partial_\nu T^{\mu\nu} \quad \& \quad p_\mu p^\mu = m^2 - \mu_0\epsilon_0\,\vec{p}^2$$

And the Lagrangian density: $\boxed{\mathcal{L} = L/vol. = -\frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu}$,

$$\text{where } J_\mu = \begin{bmatrix} \frac{\rho}{\sqrt{\mu_0\epsilon_0}} & -\vec{J} \end{bmatrix}, \text{ or: } J^\mu = \begin{bmatrix} \frac{\rho}{\sqrt{\mu_0\epsilon_0}} & \vec{J} \end{bmatrix}^T$$

Consequently, continuity equation: $\boxed{\partial_\nu J^\nu = 0}$

Variational Principle and Field Equations of Motion

$$L(q,\partial_t q) \to \mathcal{L}(A^\mu,\partial_\nu A^\mu) \Rightarrow S = \int d^4x\mathcal{L}(A^\mu,\partial_\nu A^\mu) \text{ together with } \partial_\nu(\delta A^\mu) = \delta(\partial_\nu A^\mu)$$

$$\text{as } A^\mu \to A^\mu + \delta A^\mu \overset{\partial}{\Longrightarrow} \partial_\nu A^\mu \to \partial_\nu A^\mu + \partial_\nu(\delta A^\mu) \quad \& \quad \partial_\nu A^\mu \to \partial_\nu A^\mu + \delta(\partial_\nu A^\mu)$$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial A^\mu}\delta A^\mu + \frac{\partial\mathcal{L}}{\partial[\partial_\nu A^\mu]}\delta(\partial_\nu A^\mu) = \frac{\partial\mathcal{L}}{\partial A^\mu}\delta A^\mu + \frac{\partial\mathcal{L}}{\partial[\partial_\nu A^\mu]}\partial_\nu(\delta A^\mu)$$

$$0 = \delta S \sim \int \delta\mathcal{L} - \delta\mathcal{L} \sim \int \left\{ \frac{\partial\mathcal{L}}{\partial A^\mu}\delta A^\mu - \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\nu A^\mu]}\right)\delta A^\mu + \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\nu A^\mu]}\delta A^\mu\right)\right\} - \int\delta\mathcal{L}$$

$$= \int \left\{ \left[\frac{\partial\mathcal{L}}{\partial A^\mu} - \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\nu A^\mu]}\right)\right]\delta A^\mu + \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\nu A^\mu]}\delta A^\mu\right)\right\} - \int\delta\mathcal{L}$$

$$\delta A^\mu = A^\mu(x^\lambda + \epsilon^\lambda) - A^\mu(x^\lambda) = \epsilon^\lambda\partial_\lambda A^\mu \quad \& \quad \delta\mathcal{L} = \epsilon^\lambda\partial_\lambda\mathcal{L} = \eta^{\lambda\nu}\epsilon^\lambda\partial_\nu\mathcal{L}$$

$$0 \sim \int\left[\frac{\partial\mathcal{L}}{\partial A^\mu} - \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\nu A^\mu]}\right)\right]\delta A^\mu + \int \epsilon^\lambda\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\nu A^\mu]}\partial^\lambda A^\mu - \eta^{\nu\lambda}\mathcal{L}\right) \to \text{Var.} \equiv \text{Eq.Mo.} \oplus \partial_\nu T^{\nu\lambda}$$

Every continuous symmetry generated by a non–dissipative action has a corresponding conserved quantity.

Laws of EM in "Array" Notations & Special Theory of Relativity

$$\mathcal{L}(A^\mu, \partial_\nu A^\mu) = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\nu\mu} - J^\nu A_\nu = -\frac{1}{4\mu_0} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - J^\nu A_\nu$$

Then $\boxed{\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A^\lambda - \eta^{\mu\nu} \mathcal{L} := T^{\mu\nu}} \Longrightarrow \eta^{\mu\nu} f_\nu + \partial_\nu T^{\mu\nu} = 0 \xrightarrow{f_\nu = 0} \partial_\nu T^{\mu\nu} = 0$ "Noether".

And $\boxed{\frac{\partial \mathcal{L}}{\partial A_\nu} - \left[\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right] = 0} \Longrightarrow \begin{cases} \partial_\mu F^{\mu\nu} = \mu_0 J^\nu \\ \partial_\lambda F^{\mu\nu} + \partial_\nu F^{\lambda\mu} + \partial_\mu F^{\nu\lambda} = 0 \end{cases}$

$$\partial_\lambda \eta^{\lambda\kappa} \partial_\kappa F^{\mu\nu} = \partial_\lambda \partial^\lambda F^{\mu\nu} = \Box F^{\mu\nu} = -\mu_0 \left(\partial^\mu J^\nu + \partial^\nu J^\mu\right) \xrightarrow{\text{in vacuum}}$$

$$\boxed{\partial_\lambda \partial^\lambda F^{\mu\nu} = \Box F^{\mu\nu} = 0} \Longrightarrow \mu_0 \epsilon_0 \partial_t^2 F^{\mu\nu} - \vec{\nabla}^2 F^{\mu\nu} = 0, \quad \text{Or: } \frac{1}{c^2} \frac{\partial^2}{\partial t^2} F^{\mu\nu} = \vec{\nabla}^2 F^{\mu\nu}$$

And the solution is $\boxed{F^{\mu\nu} = \mathcal{F}^{\mu\nu} e^{i(\vec{p}\vec{x} - \omega t)}}$

Consequently, $p_\mu p^\mu = m^2 - (\mu_0 \epsilon_0) \vec{p}^{\ 2} \Rightarrow \boxed{(p_\mu p^\mu) c^4 = E^2 - (\vec{p} c)^2}$

And for EM waves $p_\mu p^\mu \equiv m_0^2 = 0$. In context of QM, $p_\mu p^\mu$ called a "Casimir Operator".

Noether Symmetries & Equations of Motion

"Irreducible" Electromagnetic Waves & Lorenz Gauge

Let's study the Electric-Gauss-Ampère law $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$

$$\partial_\lambda \partial^\lambda A^\mu = \Box A^\mu = \mu_0 J^\mu - \partial^\mu(\partial_\nu A^\nu) \Longrightarrow \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - \frac{\partial^2}{\partial x^2}\vec{A} = \mu_0 \vec{J} - \vec{\nabla}(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t})$$

Helmholtz: $\vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\nabla}\theta \quad , \quad \phi \longrightarrow \phi' = \phi - \frac{1}{c^2}\partial_t \theta$ or: $\boxed{A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu \theta}$

such that $F'_{\mu\nu} = F_{\mu\nu} + \cancel{(\partial_\mu\partial_\nu\theta - \partial_\nu\partial_\mu\theta)}$

You might think $\vec{\nabla}\cdot(\delta\vec{A}) + \partial_t(\delta\phi) = \vec{\nabla}\cdot\vec{\nabla}\theta - \frac{1}{c^2}\partial_t^2\theta = \Box\theta = 0$, i.e., θ could be a field.

However, $\Box A^\mu + \partial^\mu\partial_\nu A^\nu = \mu_0 J^\mu \xrightarrow[A^\mu // p^\mu]{Fourier} J^\mu = 0 \Rightarrow A^\mu \propto \partial^\mu\theta$ is a "harmonic pure gauge"

Anyway, it seems we're "forced" to "choose" $\Box\theta = \partial_\nu A^\nu = 0$. And together with $\partial_\mu J^\mu = 0$,

we get $\Box A^\mu = \mu_0 J^\mu \xrightarrow{\text{in vacuum}} \boxed{\Box A^\mu = 0}$ with d.o.f$= 4 - 1 - 1 = 2$

Noether Symmetries & Equations of Motion

Too Abstract?!

Juan Maldacena, "The Symmetry and Simplicity of the Laws of Physics and the Higgs Boson," Eur.J.Phys. 37 (2016) 1, 015802. ArXiv: 1410.6753 [physics.pop-ph]

Thank You!